International Journal of Mathematics and Computer Science, **10**(2015), no. 1, 49–52

A Generalization of Wallis Formula

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(Received January 3, 2015, Accepted April 2, 2015)

Abstract

We generalize Wallis Formula using the Riemann zeta function.

1 Introduction

Wallis Formula is

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}...$$

Now let $z = \sigma + it$. For $\sigma > 1$, the **Riemann zeta function** ζ is defined by $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$.

We have $|\frac{1}{n^z}| = \frac{1}{|e^{z \log n}|} = \frac{1}{|e^{\sigma \log n}|} = \frac{1}{n^\sigma}$. By Weierstrass test, the series $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges uniformly in the half-plane $\sigma > 1$ and hence on every compact subset of this half-plane. Thus ζ is analytic in the half-plane $\sigma > 1$ being the sum function of a uniformly convergent series of analytic functions. With some work, this function can be continued analytically to all complex $z \neq 1$. As a result, the zeta function is analytic everywhere except for a simple pole at $z = 1$ with residue 1. It is well-known that the only real zeros of the zeta function are on the negative even integers and are called trivial zeros.

When talking about the zeta function, it would be a miss not to mention the following famous conjecture:

The Riemann Hypothesis. [3] ALL NON-TRIVIAL ZEROS of the zeta function have real part equal to $\frac{1}{2}$.

Key words and phrases: Wallis Formula, Zeta function.
AMS (MOS) Subject Classifications: 30D05, 30D15, 11M06, 11M38, 11M50.
ISSN 1814-0432, 2015, http://ijmcs.future-in-tech.net

The **completed zeta function** (or generically the xi-function), originally defined by Riemann [3], is

$$\xi(z) = \frac{1}{2}z(z-1)\pi^{-\frac{z}{2}}\Gamma(\frac{z}{2})\zeta(z)$$

The success of our proof (generalization) hinges on first finding a product representation of the completed zeta function which then is used in finding a product representation of the zeta function. To have a relatively self-contained paper we mention a few definitions:

Let $f(z)$ be an entire function. The **maximum modulus function**, denoted by $M(r)$, is defined by $M(r) = \max\{|f(z)| : |z| = r\}$.

Let $f(z)$ be a non-constant entire function. The **order** ρ of $f(z)$ is defined by

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

The order of any constant function is 0, by convention.

An entire function $f(z)$ of positive order ρ is said to be of **type** τ if

$$\tau = \limsup_{r\to\infty} \frac{\log M(r)}{r^\rho}.$$

The following [1] are some important properties of the completed zeta function:

1. $\xi(z) = \xi(1-z)$. This Functional Equation shows that the function $\xi(z)$ is symmetric about the critical line $Re(z) = \frac{1}{2}$.
2. The function $\xi(z)$ is entire.
3. The function $\xi(z)$ is of order one and infinite type.
4. The function $\xi(z)$ has infinitely many zeros.

Remark 1.1. *It is clear now that the completed zeta function ξ is more convenient to use instead of the zeta function ζ since using the definition of ξ removes the simple pole of ζ at $z = 1$ and as a result the theory of entire functions can be applied if needed, to ξ. In addition, since none of the factors of ξ except ζ has a zero in $\mathbb{C} - \{0, 1\}$, no information is lost about the non-trivial zeros.*

The Riemann Hypothesis can therefore be stated as:
The Riemann Hypothesis using the completed zeta function. ALL ZEROS of $\xi(z)$ are on the critical line $Re(z) = \frac{1}{2}$.

2 Main Theorem

We begin by finding a canonical representation of $\xi(z)$:
$\xi(z)$ is an entire function of order one and infinite type. Since the zeta function $\zeta(z)$ has a simple pole with residue 1 at $z = 1, \xi(1) = \frac{1}{2}$. Now, using the functional equation $\xi(z) = \xi(1-z), \xi(0) = \frac{1}{2}$ Then, by ([2] p. 47), we have

$$\xi(z) = e^A e^{Bz} \prod_{n=1}^{\infty} (1 - \frac{z}{z_n}) \exp \frac{z}{z_n},$$

where again $\{z_n\}_1^\infty$ are the non-zero zeros of $\xi(z)$ and hence, using the functional equation and the definition of the completed zeta function, are the non-trivial zeros of $\zeta(z)$ which are indeed in the critical strip and A and B are complex constants.
Now $\xi(0) = \frac{1}{2}$ implies that $e^A = \frac{1}{2}$ and so we can write

$$\xi(z) = \frac{1}{2} e^{Bz} \prod_{n=1}^{\infty} (1 - \frac{z}{z_n}) \exp \frac{z}{z_n}.$$

The effort to find B using $\xi(1) = \frac{1}{2}$ leads to

$$1 = e^b \prod_{n=1}^{\infty} (1 - \frac{1}{z_n}) e^{\frac{1}{z_n}}.$$

Consider the product

$$p = \prod_{n=1}^{\infty} (1 - \frac{1}{z_n}) e^{\frac{1}{z_n}}.$$

Then

$$p^z = \prod_{n=1}^{\infty} (1 - \frac{1}{z_n})^z e^{\frac{z}{z_n}}.$$

Therefore,

$$\xi(z) = \frac{1}{2} e^{Bz} p^z \prod_{n=1}^{\infty} (1 - \frac{z}{z_n})(1 - \frac{1}{z_n})^{-z}.$$

Now $\xi(1) = \frac{1}{2}$ implies that $e^B p = 1$ and our identity reduces to the following representation of $\xi(z)$:

$$\xi(z) = \frac{1}{2} \prod_{n=1}^{\infty} (1 - \frac{z}{z_n})(1 - \frac{1}{z_n})^{-z}.$$

Using the product representations of $\xi(z)$ and $\frac{1}{\Gamma(z)}$ with the definition of $\xi(z)$ give the following explicit representation for $\zeta(z)(\{z_n\}$ denotes the sequence of non-trivial zeros of $\zeta(z))$:

$$\zeta(z) = \underbrace{\frac{1}{z-1}}_{\text{for singularity}} \pi^{\frac{z}{2}} \prod_{n=1}^{\infty} \underbrace{(1 + \frac{z}{2n})}_{\text{for trivial zeros}} \underbrace{(1 - \frac{z}{z_n})}_{\text{for non-trivial zeros}} (1 + \frac{1}{n})^{-\frac{z}{2}} (1 - \frac{1}{z_n})^{-z}.$$

3 Wallis Formula as an easy corollary

Our discovered explicit representation above of the zeta function serves as a generalization of Wallis Formula as the following corollary shows:

Corollary 3.1. Wallis Formula

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}...$$

Proof. For $z \in \mathbb{C} - \{0, 1\}$, we can rewrite the representation in the theorem as:

$$(z-1)\zeta(z) = \frac{\pi^{\frac{z}{2}}}{\frac{z}{2}} \underbrace{\prod_{n=1}^{\infty}(1-\frac{z}{z_n})(1-\frac{1}{z_n})^{-z}}_{2\xi(z) \text{ entire hence continuous}} \underbrace{\frac{z}{2}\prod_{n=1}^{\infty}(1+\frac{z}{2n})(1+\frac{1}{n})^{-\frac{z}{2}}}_{\frac{1}{\Gamma(\frac{z}{2})} \text{ entire hence continuous}}.$$

The result now follows using $\lim_{z\to 1}(z-1)\zeta(z) = 1$.

References

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