

Half a dozen famous unsolved problems in mathematics with a dozen suggestions on how to try to solve them

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Abstract

Littlewood [17] stated "Try a hard problem. You may not solve it, but you will solve something else". In this paper, we concentrate on six of the most famous and important unsolved problems in Mathematics. For each such problem we offer two suggestions to try to solve it. Even though some of these suggestions have been tried by the author with interesting results, he hopes that others will have more success thus attaining the objective of advancing knowledge. Our first problem begins with (clear) prime numbers while our last problem concludes with (hidden) prime numbers (Read on to discover the difference).

1 Introduction

A few mathematicians have tried hard problems with the intention to solve them and few have succeeded. Among these was Andrew Wiles who settled Fermat's Last Theorem after working on it for seven years while he was a Professor at Princeton. He did so after promising at age 10 that he [33] "would be the first person to solve it"; (Author's comment: second if you

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believe Fermat statement: "I have a truly marvelous proof of this proposition which this margin is too narrow to contain". I don't personally believe it unless someone rediscovers it to beat Wiles over one hundred pages proof. After all, in his first paper on Number Theory in 1732 [9], Euler factored Fermat's number F_5 as $(741)(6700417)$ refuting Fermat Conjecture that all Fermat's Numbers $F_n = 2^{2^n} + 1$ are prime. This consolidates the power and uniqueness of mathematics that does not accommodate stereotyping).

I belong to the group of mathematicians who have adopted the "Littlewood Strategy" in a portion of my research and, I believe, that is a large group as is evidenced in the enormous amount of research resulting from working on hard problems that will fill up a million volumes, if not more, and have opened up new fields of mathematics.

Taking up this strategy the author, for many years, has looked at some of these six hard problems and indeed, obtained some results. But, of course, more needs to be done and, hopefully, this article will serve as an impetus to further research. Therefore, the objective of this article is not only to highlight the modest work of the author but also to offer this list of my favorite six problems among a wish-to-be-solved list for the 21st Century, whatever that is or may be.

We present these problems using the order from old to more recent. Therefore, in section 2 we consider the twin prime problem (Conjecture) while in section 3 we consider the Goldbach Conjecture. In section 4, we deal with (the one-million dollar if proven or disproven) Riemann Hypothesis. Section 5 is about the zeta function at odd arguments, principally 3 while section 6 is about the relatively recent (with another one-million dollar if proven or disproven) Beal Conjecture. In section 7, we conclude with challenge numbers that play a role in the Rivest, Shamir, Adleman (RSA) Cryptosystem [31].

2 Twin Prime Conjecture

We need the following:

Definition 2.1. Let $f(z)$ be an entire function. The maximum modulus function, denoted by $M(r)$, is defined by $M(r) = \max \{|f(z)| : |z| = r\}$.

Definition 2.2. Let $f(z)$ be a non-constant entire function. The order ρ of $f(z)$ is defined by

$$\rho = \limsup_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r}.$$

The order of any constant function is 0, by convention.

Definition 2.3. An entire function $f(z)$ of positive order ρ is said to be of type τ if

$$\tau = \limsup_{r \rightarrow \infty} \frac{\log M(r)}{r^\rho}.$$

Now the number of Primes 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \dots is infinite. However, no one knows, whether the number of Twin Primes (3,5), (5,7), (11,13), (17,19), \dots , (2996863034895.2¹²⁹⁰⁰⁰⁰-1, 2996863034895.2¹²⁹⁰⁰⁰⁰+1), and so on is finite or infinite.

The Twin Prime Conjecture, TPC, states that the number of Twin Primes is infinite.

Brun [3](in a failed attempt to settle TPC) showed that:

With p_n and $p_n + 2$ being prime numbers, $\sum_{n=1}^{\infty} \frac{1}{p_n}$ converges.

More than 50 years later, the sum $\sum_{n=1}^{\infty} (\frac{1}{p_n} + \frac{1}{p_n+2})$ known as Brun's constant B has been calculated by Shanks and Wrench [32] and by Brent [18] to be approximately 1.90216054.

Using Brun's result, I have shown [10] that TPC equivalent to: Order of $\prod(1 - \frac{z}{p_n}) = 1$. (I have also shown that Order of $\prod(1 - \frac{z}{p_n}) \leq 1$ and Order of $\prod(1 - \frac{z}{p_n})e^{\frac{z}{p_n}} = 1$).

My first suggestion is, of course, to try to show Order of $\prod(1 - \frac{z}{p_n}) \geq 1$ or that the Order of $\prod(1 - \frac{z}{p_n})$ cannot be less than 1.

As a prerequisite to my second suggestion let us give a definition and state a theorem (for a proof see [24], Theorem 12, p. 22):

Definition 2.4. If $0 \leq \tau < \infty$, then the entire function $f(z)$ is said to be of **finite type**.

If $\tau = 0$, then $f(z)$ is said to be of **minimal type**.

If $0 < \tau < \infty$, then $f(z)$ is said to be of **normal type**.

If $\tau = \infty$, then $f(z)$ is said to be of **infinite or maximal type**.

Theorem 2.5.

a) The order of the product of two entire functions of different orders is equal to the larger of the orders of the factors, and the type is equal to the type of the function that has the larger order.

b) *The product of two entire functions of the same order, one having normal type σ and the other having minimal type, is an entire function of the same order and type σ .*

c) *The product of two entire functions of the same order ρ , one having maximal type and the other having at most normal type, is an entire function of order ρ and maximal type.*

Now my second suggestion revolves about trying to multiply $\prod(1 - \frac{z}{p_n})e^{\frac{z}{p_n}}$ by the appropriate entire function to show that Order of $\prod(1 - \frac{z}{p_n}) = 1$ (but this has not been yielding to me as the natural entire function to use is e^{-Bz} which is of order 1 and type B so one needs a theorem to cover products of entire functions of the same order and the same type as the Type of $\prod(1 - \frac{z}{p_n})e^{\frac{z}{p_n}}$ turns out [10] to be B).

3 Goldbach Conjecture

In a letter to Euler in 1742, Christian Goldbach conjectured that: Every natural number $n > 5$ is a sum of three primes.

Euler restated the conjecture as: Every EVEN number ≥ 4 is the sum of two primes.

In 1919 using his Sieve Method, Brun [4] proved that every large EVEN number is the sum of two numbers each having at most 9 prime factors. This result was slightly improved by many authors afterwards.

In 1947, Rényi [29] proved that there is a positive integer M such that every sufficiently large EVEN integer is the sum of a prime p and a number A , where A has no more than M prime factors. This important result was refined by many who gave explicit values of M but, by so far, the best result was by Jingrun Chen [5] who, after improvements to Brun's method, showed in 1966 that:

Every sufficiently large EVEN integer is the sum of a prime and the product of at most two primes (due to the "parallelism" between the Twin Prime Conjecture and Goldbach Conjecture, it is noteworthy here that Chen proved at the same time a result close to the Twin Prime Conjecture that there are infinitely many primes p such that $p + 2$ is a prime or is a product of at most two primes and how about the summer of 2013 where the two breakthroughs

by Zhang [36] and Helfgott [19] happened, the first being related to the Twin Prime Conjecture while the second to the Goldbach Conjecture-case of odd integers).

In 1923 an important result by Hardy and Littlewood stated that under the Generalized Riemann Hypothesis, every large ODD integer is the sum of three primes.

In 1937, Vinogradov improved Hardy-Littlewood Theorem by proving the following important theorem:

Theorem 3.1. *(Vinogradov Theorem) [35] Every sufficiently large ODD number $n > n_o$ is the sum of three primes.*

However, an elaborate study of the proof by Borodzkin (Vinogradov student) [2] revealed that "sufficiently large" involves $n_o = 3^{3^{15}}$ which is approximately $10^{7000000}$. Even though the above estimate was refined by Chen (the same Chen) and Wang (Chen student)[6] to $10^{43000} \sim e^{99012}$ and later by Liu (another Chen student) and Wang [25] to e^{3100} , it was still too large for cases to be checked by computer.

On the other hand, the following is a surprising and relatively recent result (May 13, 2013) by Harald Helfgott [19]:

Theorem 3.2. *Every ODD number > 5 is the sum of 3 primes.*

Indeed, this result was proven, by Helfgott, for odd numbers $> (8.875)10^{30}$ but the result has been checked by Platt [20] by computer for odd numbers $< (8.875)10^{30}$.

Remark 3.3. *It is interesting to note here that Vinogradov has succeeded in proving this theorem without assuming the Generalized Riemann Hypothesis which Hardy and Littlewood had proved earlier and now Helfgott has succeeded in the continuous effort to prove Vinogradov Theorem without the assumption of "sufficiently large".*

In 1938, Estermann [8] and Tchudakov [34] proved that almost all EVEN integers are sums of two primes which is strengthened in 2013 by a result of Oliveira e Silva, Herzog, and Pardi [27] who have verified using computers that every even integer up to $(4)10^{18}$ is the sum of two primes, one suggestion

is to strengthen Chen's theorem to show that every large EVEN integer is the sum of two primes.

Using the previous observations, another suggestion is to explicitly find a positive integer n_0 such that every EVEN number $n > n_0$ is the sum of two primes (presumably $n_0 > (4)10^{18}$) and then check that the desired result holds for all even numbers m with $(4)10^{18} < m \leq n_0$ using a (super)computer or many (super)computers working in parallel.

4 Riemann Hypothesis

Let $z = \sigma + it$. For $\sigma > 1$, the **Riemann zeta function** ζ is defined by $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$. Now $|\frac{1}{n^z}| = \frac{1}{|e^{z \log n}|} = \frac{1}{|e^{\sigma \log n}|} = \frac{1}{n^\sigma}$. By Weierstrass test, the series $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges uniformly in the half-plane $\sigma > 1$ and hence on every compact subset of this half-plane. Thus ζ is analytic in the half-plane $\sigma > 1$ being the sum function of a uniformly convergent series of analytic functions. This function can be continued analytically to all complex $z \neq 1$ as follows:

$$(1 - 2^{1-z})\zeta(z) = (1 - 2 \cdot \frac{1}{2^z})(1 + \frac{1}{2^z} + \frac{1}{3^z} + \dots) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^z}.$$

Therefore,

$$\zeta(z) = \frac{1}{1 - 2^{1-z}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^z}$$

which is analytic on $\sigma > 0$ except $z = 1$ as the other values of z for which $1 - 2^{1-z} = 0$ are removable. To continue (extend) this to $\sigma < 1$, we use Riemann Functional Equation, $\zeta(z) = 2^z \pi^{z-1} \sin \frac{\pi z}{2} \Gamma(1-z) \zeta(1-z)$. As a result, the zeta function is analytic everywhere except for a simple pole at $z = 1$ with residue 1.

Considering Riemann Functional Equation $\zeta(z) = 2^z \pi^{z-1} \sin \frac{\pi z}{2} \Gamma(1-z) \zeta(1-z)$ again, multiplying by $1-z$, using $(1-z)\Gamma(1-z) = \Gamma(2-z)$, and taking the limit as z tends to 1, we get $\zeta(0) = -\frac{1}{2}$.

Bernhard Riemann's 1859 paper [30] had a deep impact on Analytic Number Theory of which we mention $\prod_p \text{prime} \frac{1}{1 - \frac{1}{p^z}} = \sum_{n=1}^{\infty} \frac{1}{n^z}$ previously discovered by Euler but for real z . In other words, the Riemann zeta function is not only

important as a function of a complex variable but also contains information about prime numbers and their distribution.

Riemann's defined zeta function can now be related to Bernoulli numbers $B_k, k = 0, 1, 2, 3, \dots$ defined by $\frac{z}{e^z - 1} = \sum_0^\infty B_k \frac{z^k}{k!}$ as in Euler famous identity [13]:

$$\zeta(2k) = \frac{(-1)^{k-1} 2^{2k-1} B_{2k} \pi^{2k}}{(2k)!}$$

and [11] if k is a positive integer, then $\zeta(-k) = -\frac{B_{k+1}}{k+1}$. In particular, the latter case implies that $\zeta(-2k) = 0$ (see [12], p. 21 for a proof). The negative even integers are called **trivial zeros** of the zeta function. The other zeros (there are plenty as we shall see later) are called the **non-trivial zeros** and we'll show that they are confined INSIDE what is known as the **Critical Strip** $\{z : 0 \leq \operatorname{Re} z \leq 1\}$. To see this, first note that the Euler Product Formula implies that there are no zeros of $\zeta(z)$ with real part > 1 since convergent infinite products never vanish. Next, using Riemann Functional Equation $\pi^{-\frac{z}{2}} \Gamma(\frac{z}{2}) \zeta(z) = \pi^{-\frac{1-z}{2}} \Gamma(\frac{1-z}{2}) \zeta(1-z)$ and the fact that Γ has no zeros in \mathbb{C} , it follows that there are no zeros of $\zeta(z)$ with real part < 0 apart from $\dots, -6, -4, -2$. Finally, using $\zeta(1+it) \neq 0, \forall t \in \mathbb{R}$ and the Functional Equation again, the result follows.

In the same paper Riemann conjectured that ALL non-trivial zeros of his zeta function have real part equal to $\frac{1}{2}$. This has been known as the **Riemann Hypothesis**.

Definition 4.1. *The **completed zeta function** (or generically the xi-function), originally defined by Riemann [30], is*

$$\xi(z) = \frac{1}{2} z(z-1) \pi^{-\frac{z}{2}} \Gamma(\frac{z}{2}) \zeta(z)$$

(Mathematically, the Gamma function was needed to complete the zeta function. Symbolically, ζ zeta; ξ completed).

Theorem 4.2. [11] *The following are some important properties of the completed zeta function:*

1. $\xi(z) = \xi(1-z)$. This Functional Equation shows that the function $\xi(z)$ is symmetric about the critical line $\operatorname{Re}(z) = \frac{1}{2}$.
2. The function $\xi(z)$ is entire.

- 3. The function $\xi(z)$ is of order one and infinite type.
- 4. The function $\xi(z)$ has infinitely many zeros.

Remark 4.3. It is clear now that the completed zeta function ξ is more convenient to use instead of the zeta function ζ since using the definition of ξ removes the simple pole of ζ at $z = 1$ and as a result the theory of entire functions can be applied, if needed, to ξ (Property 2 in the preceding theorem). In addition, since none of the factors of ξ except ζ has a zero in $\mathbb{C} - \{0, 1\}$, no information is lost about the non-trivial zeros.

The Riemann Hypothesis can therefore be stated as:

The Riemann Hypothesis using the completed zeta function. ALL zeros of $\xi(z)$ are on the critical line $Re(z) = \frac{1}{2}$. In [14], I have obtained the following representation of $\xi(z)$:

$$\xi(z) = \frac{1}{2} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \left(1 - \frac{1}{z_n}\right)^{-z}.$$

which when used with that of $\frac{1}{\Gamma(z)}$ [13] yields the following explicit representation for $\zeta(z)$ ($\{z_n\}$ denotes the sequence of non-trivial zeros of $\zeta(z)$) :

Theorem 4.4.

$$\zeta(z) = \underbrace{\frac{1}{z-1}}_{\text{for singularity}} \pi^{\frac{z}{2}} \prod_{n=1}^{\infty} \underbrace{\left(1 + \frac{z}{2n}\right)}_{\text{for trivial zeros}} \underbrace{\left(1 - \frac{z}{z_n}\right)}_{\text{for non-trivial zeros}} \left(1 + \frac{1}{n}\right)^{-\frac{z}{2}} \left(1 - \frac{1}{z_n}\right)^{-z}.$$

Corollary 4.5. Wallis Product

$$\frac{\pi}{2} = \frac{2}{1} \frac{2}{3} \frac{4}{3} \frac{4}{5} \frac{6}{5} \dots$$

Proof. For $z \in \mathbb{C} - \{0, 1\}$, we can rewrite the representation in the theorem as:

$$(z-1)\zeta(z) = \frac{\pi^{\frac{z}{2}}}{\frac{z}{2}} \underbrace{\prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \left(1 - \frac{1}{z_n}\right)^{-z}}_{2\xi(z) \text{ entire hence continuous}} \underbrace{\frac{z}{2} \prod_{n=1}^{\infty} \left(1 + \frac{z}{2n}\right) \left(1 + \frac{1}{n}\right)^{-\frac{z}{2}}}_{\frac{1}{\Gamma(\frac{z}{2})} \text{ entire hence continuous}}.$$

The result now follows using $\lim_{z \rightarrow 1} (z-1)\zeta(z) = 1$.

For a change, let me offer somebody's else suggestions. Ricardo Marco [26] stated a couple of recommendations to use with the Riemann Hypothesis:

· A technique that allows the Euler product in the critical strip seems unavoidable. For example, a proxy for the Euler product that has a global meaning.

· Heavy Fourier analysis mixed with complex variable must be used in a non-trivial form that incorporates in full the basic arithmetic aspects."

5 Zeta function at odd arguments

No exact value of the zeta function at any odd integer ≥ 3 is known (unlike the ones at even integers ≥ 2 due to Euler which we mentioned before). Here we specialize by considering the zeta function at 3. Thus the exact value that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges to is one of the most notorious problems that did not even yield to Euler.

In 1998 [15], I have obtained the following representation

$$\zeta(3) = -\frac{\sqrt{3}}{18}\pi^3 + \frac{3\sqrt{3}}{4}\pi \sum_1^{\infty} \frac{1}{(3n-2)^2} - \frac{3}{4} \sum_1^{\infty} \frac{1}{n^3 \binom{2n}{n}}$$

One suggestion is to use the LLL algorithm in the Computer Algebra System Maple to find an integer relation which is then proved mathematically.

On the other hand, quite recently, I have discovered something interesting that is worth sharing:

Write Wallis Formula as

$$\begin{aligned} \frac{\pi}{2} &= \frac{2.2}{1.3} \frac{4.4}{3.5} \frac{6.6}{5.7} \dots = \prod_1^{\infty} \frac{(2n)(2n)}{(2n-1)(2n+1)} = \prod_1^{\infty} \frac{(2n-1+1)(2n+1-1)}{(2n-1)(2n+1)} \\ &= \prod_1^{\infty} \left(1 + \frac{1}{2n-1}\right) \left(1 - \frac{1}{2n+1}\right) = \prod_1^{\infty} \left(1 - \frac{1}{4n^2-1}\right). \end{aligned}$$

Using partial fractions

$$\sum_1^{\infty} \frac{1}{4n^2-1} = \lim_{k \rightarrow \infty} \sum_1^k \left[\frac{\frac{1}{2}}{2k-1} - \frac{\frac{1}{2}}{2k+1} \right] = \lim_{k \rightarrow \infty} \frac{1}{2} \left(1 - \frac{1}{2k+1}\right) = \frac{1}{2}.$$

We see that $\pi \sum_1^{\infty} \frac{1}{4n^2-1} = \prod_1^{\infty} \left(1 - \frac{1}{4n^2-1}\right)$. and so the fraction $\frac{1}{4n^2-1}$ is interesting.

It is natural to look now at $\sum_1^\infty \frac{1}{n(4n^2-1)}$. We found out that this requires the following lemma:

Lemma 5.1. $\lim_{k \rightarrow \infty} (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2k-1} - \frac{1}{2} \ln k - \frac{1}{2} \ln 4) = \frac{1}{2}\gamma$, where γ is Euler's Constant.

Proof. We use the well-known formula $\lim_{n \rightarrow \infty} (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n) = \gamma$. For the first case $n = 2k + 1$ we have

$$\lim_{k \rightarrow \infty} (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2k+1} - \ln(2k+1)) = \gamma.$$

So

$$\lim_{k \rightarrow \infty} (1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2k-1} + [\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2k}] + \frac{1}{2k+1} - \ln(2k+1)) = \gamma$$

which we write as

$$\lim_{k \rightarrow \infty} (1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2k-1} - \frac{1}{2} \ln k + \frac{1}{2} \ln k - \frac{1}{2} \ln 4 + \frac{1}{2} \ln 4 + \frac{1}{2} [1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}] + \frac{1}{2k+1} - \ln(2k+1)) = \gamma$$

or

$$\lim_{k \rightarrow \infty} (1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2k-1} - \frac{1}{2} \ln k + \ln k - \frac{1}{2} \ln 4 + \frac{1}{2} \ln 4 + \frac{1}{2} [1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} - \ln k] + \frac{1}{2k+1} - \ln(2k+1)) = \gamma.$$

The desired result follows because

$$\lim_{k \rightarrow \infty} (\ln k + \frac{1}{2k+1} - \ln(2k+1) + \frac{1}{2} \ln 4) = \lim_{k \rightarrow \infty} (\ln \frac{2k}{2k+1} + \frac{1}{2k+1}) = 0.$$

For the second case $n = 2k$ we have

$$\lim_{k \rightarrow \infty} (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2k} - \ln(2k)) = \gamma.$$

Then

$$\lim_{k \rightarrow \infty} (1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2k-1} + [\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2k}] - \ln(2k)) = \gamma.$$

Hence

$$\lim_{k \rightarrow \infty} \left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2k-1} - \frac{1}{2} \ln k + \frac{1}{2} \ln k - \frac{1}{2} \ln 4 + \frac{1}{2} \ln 4 + \frac{1}{2} \left[1 + \frac{1}{2} + \dots + \frac{1}{k}\right] - \ln(2k)\right) = \gamma.$$

Therefore

$$\lim_{k \rightarrow \infty} \left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2k-1} - \frac{1}{2} \ln k - \frac{1}{2} \ln 4 + \frac{1}{2} \left(1 + \frac{1}{2} + \dots + \frac{1}{k} - \ln k\right)\right) = \gamma$$

from which the result follows.

Theorem 5.2.

$$\sum_1^{\infty} \frac{1}{n(4n^2-1)} = -1 + \ln 4$$

Proof. We have

$$\frac{1}{n(4n^2-1)} = \frac{-1}{n} + \frac{1}{2n-1} + \frac{1}{2n+1}.$$

So

$$\begin{aligned} \sum_1^{\infty} \frac{1}{n(4n^2-1)} &= \sum_1^{\infty} \left(\frac{-1}{n} + \frac{1}{2n-1} + \frac{1}{2n+1}\right) = \lim_{k \rightarrow \infty} \sum_1^k \left(\frac{-1}{n} + \frac{1}{2n-1} + \frac{1}{2n+1}\right) \\ &= \lim_{k \rightarrow \infty} \left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2k-1} + \left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2k-1} + \frac{1}{2k+1}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}\right)\right) \\ &= \lim_{k \rightarrow \infty} \left(-1 + 2\left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2k-1}\right) + \frac{1}{2k+1} - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}\right)\right) \\ &= \lim_{k \rightarrow \infty} \left(-1 + 2\left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2k-1}\right) - \ln k + \ln k + \frac{1}{2k+1} - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}\right)\right) \\ &= \lim_{k \rightarrow \infty} \left(-\frac{2k}{2k+1} + 2\left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2k-1} - \frac{1}{2} \ln k - \frac{1}{2} \ln 4\right) + \ln 4 + \left(\ln k - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}\right)\right)\right) \\ &= -1 + \gamma - \gamma + \ln 4 = -1 + \ln 4. \end{aligned}$$

Moreover,

$$\sum_1^{\infty} \frac{1}{n^2(4n^2-1)} = \sum_1^{\infty} \left[\frac{-1}{n^2} + \frac{4}{4n^2-1}\right] = -\frac{\pi^2}{6} + 2,$$

Furthermore,

$$\sum_1^{\infty} \frac{1}{n^3(4n^2-1)} = \sum_1^{\infty} \left[\frac{-1}{n^3} + \frac{4}{n(4n^2-1)}\right] = -\zeta(3) + 4(-1 + \ln 4),$$

Consequently,

$$\zeta(3) = 4(-1 + \ln 4) - \sum_1^{\infty} \frac{1}{n^3(4n^2 - 1)}.$$

So another suggestion pops up and that is to use the latter series, as a means, to find the exact value of $\zeta(3)$.

6 Beal Conjecture

A relatively new (1980s) conjecture, made by the amateur mathematician Andrew Beal [1] (Beal who is a successful banker has recently increased his prize to a million dollars for its proof or disproof and the prize is actually held by the American Mathematical Society), states that the only solutions to the equation $a^x + b^y = c^z$, when a, b and c are positive integers, and x, y and z are positive integers greater than two, are those in which a, b and c have a common prime factor.

Taking the contrapositive, Beal Conjecture can now be restated as:

Suppose x, y and z are positive integers greater than two. Then the equation $a^x + b^y = c^z$, has no solution when a, b and c are positive integers that are coprime.

This reformulation shows the importance of the conjecture due to its relevance to a similar but much older famous problem, Fermat's Last Theorem, which states that with $n > 2$, there are no nonzero integers a, b, c such that $a^n + b^n = c^n$, which was proved by Andrew Wiles [33] (Wiles first announced his proof of this monumental theorem on June 23, 1993 at a lecture in Cambridge carefully entitled "Elliptic Curves and Galois Representations" to maintain secrecy).

Remark 6.1. *What made Beal conjecture famous in the eighties is that it is stronger than Fermat's Last Theorem: the truth of the Beal Conjecture implies Fermat's Last Theorem, but not conversely.*

If you believe the conjecture is true, then examine Wiles 108-page paper thoroughly and go from there.

On September 22, 1994, Professor Harold M. Edwards, an authoritative figure on Fermat's Last Theorem [7], wrote Beal "It would be quite remarkable if your generalization of Fermat's last theorem were true. I suspect, though, that a serious computer search would turn up an example in which

the equation that you say is impossible holds.” Like Edwards, if you believe the conjecture is false, then get yourself a supercomputer and try to find a counterexample.

7 RSA Challenge Numbers

In general, it is easy to multiply and, indeed, there are machines that multiply two extremely large (prime) numbers in seconds!

Factoring, however, is not always easy and below are some challenge numbers with prizes for finding their TWO prime factors:

RSA-896: 412023436986659543855531365332575948179816998443279828
4545562643387644556524842619809887042316184187926142024718886949
256093177637503342113098239748515094490910691026986103186
27041148808669705649029036536588674337317208131041051
90864254793282601391257624033946373269391
(number of digits=270). Prize=75 grand.

RSA-1024: 13506641086599522334960321627880596993888147560566702752
4485143851526510604859533833940287150571909441798207282164
47155137368041970396419174304649658927425623934102086438
58509643110564073501508187510676532021103729587257623946
29205563685529475213500852879416377328533906109750544
334999811150056977236890927563
(number of digits=309). Prize=100 grand.

RSA-1536: 1847699703211741474306835620200164403018
54933866341017147178577491065169671116124985933768
43054357445856160615445717940522297177325246609606
46946071249623720442022269756756687378427562389508
76467844093328515749657884341508847552829818672645
13398633649319080846719904318743812833635027954702
82653297802934916155811881049844908319545009848393
77522725705257859194499387007369575568843693381277
96130892303925696952532616208236764903160365513714
7913932347169566988069
(number of digits=463). Prize=150 grand.

RSA-2048: 2519590847565789349402718324004839857142

92821262040320277771378360436620207075955562640185
 25880784406918290641249515082189298559149176184502
 80848912007284499268739280728777673597141834727026
 18963750149718246911650776133798590957000973304597
 48808428401797429100642458691817195118746121515172
 65463228221686998754918242243363725908514186546204
 35767984233871847744479207399342365848238242811981
 63815010674810451660377306056201619676256133844143
 60383390441495263443219011465754445417842402092461
 65157233507787077498171257724679629263863563732899
 12154831438167899885040445364023527381951378636564
 391212010397122822120720357

(number of digits=617). Prize=200 grand.

One suggestion is to imitate what was done in [23] when, in 1994,
RSA-129=11438162575788886766923577997614661201021829672124236256
 25618429357069352457338978305971235639587050589890751475992900268
 79543541

was factored as the product of

3490529510847650949147849619903898133417764638493387843990820577

and

32769132993266709549961988190834461413177642967992942539798288533

or [21] when, on December 12, 2009,

RSA-768= 1230186684530117755130494958384962720772853569595334792197322452
 1517264005072636575187452021997864693899564749427740638459251925
 5732630345373154826850791702612214291346167042921431160222124047
 9274737794080665351419597459856902143413

was factored into the product of two 116-digit prime numbers:

33478071698956898786044169848212690817704794983713768568912

431388982883793878002287614711652531743087737814467999489

and

3674604366679959042824463379962795263227915816434308764267

6032283815739666511279233373417143396810270092798736308917

My second suggestion is that if you find the first prime number, contact me and we'll definitely split the prize (just kidding; trying to trick you here by suggesting that I did half the work as well).

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