International Journal of Mathematics and
Computer Science, **4**(2009), no. 1, 21–30

The Value of the Zeta Function at an Odd Argument

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Abstract

For over 300 years the values of the zeta function at odd integers greater than or equal to 3 have remained a mystery. The PSLQ algorithm which is implemented in the Computer Algebra System Maple is considered one of the top ten algorithms of the 20th Century. We employ PSLQ to discover an Euler-type identity for such an odd argument.

1 Introduction

The function $\frac{z}{e^z-1}$ clearly has its nearest singularities at $z = -2\pi i$ and $z = 2\pi i$ and so is analytic in the disk $|z| < 2\pi$. Therefore we can represent it there as $\frac{z}{e^z-1} = \sum\limits_{0}^{\infty} B_n \frac{z^n}{n!}$. In the coefficients $B_n's$ are nothing but the Bernoulli numbers the first of which are $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, B_7 = 0, B_8 = -\frac{1}{30}, B_9 = 0$ and $B_{10} = \frac{5}{66}$. Now let us prove, in general, that $B_{2k+1} = 0$ for $k = 1, 2, 3, ...$ First, $\frac{t}{e^t-1} + \frac{t}{2} = \frac{t}{2}\coth\frac{t}{2}$ is an even function. Next, $\frac{t}{2}\coth\frac{t}{2} = \sum_0^\infty B_n \frac{t^n}{n!} + \frac{t}{2} = \sum_{n=0, n\neq 1}^\infty B_n \frac{t^n}{n!}$. Now replacing t with $-t$ we get $\frac{t}{2}\coth\frac{t}{2} = \sum_{n=0,n\neq 1}^\infty (-1)^n B_n \frac{t^n}{n!}$. Therefore, $(-1)^n B_n = B_n$ for $n = 0, 2, 3, 4, ...$ which for odd $n = 2k+1$ implies the result. The **Riemann Zeta function** ζ is defined by $\zeta(z) = \sum\limits_{k=1}^{\infty} \frac{1}{k^z}$. Set $z = x + iy$. Now $|\frac{1}{k^z}| = \frac{1}{|e^{z\log k}|} = \frac{1}{|e^{x\log k}|} = \frac{1}{k^x}$. So the series $\sum\limits_{k=1}^{\infty} \frac{1}{k^z}$ converges absolutely in the half-plane $x > 1$. Moreover, by Weierstrass test, this series

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Key words and phrases: Zeta function, Euler, Integer relations, PSLQ Algorithm, Maple.
AMS (MOS) Subject Classifications: Primary 11M06, 68W30; Secondary 11G55, 11B68.
ISSN 1814-0424

converges uniformly on every compact subset of this half-plane. This function can be continued analytically to all complex $z \neq 1$ (For $z = 1$, the series is the harmonic series which diverges to infinity. As a result, the zeta function is a meromorphic function of the complex variable z, which is analytic everywhere except for a simple pole at $z = 1$ with residue 1.)
For completeness we mention the known results $\zeta(0) = -\frac{1}{2}$ and $\zeta(-n) = -\frac{B_{n+1}}{n+1}$ for natural numbers n. In particular, $\zeta(-2n) = 0$ for natural numbers n, (these are called the trivial zeros of the zeta function) and $\zeta(1-2n) = -\frac{B_{2n}}{2n}$. Moreover, as an aside, we state the following interesting result [5] : $\zeta(\frac{1}{2}) = (\sqrt{2}+1)\sum_1^\infty \frac{(-1)^n}{n^{\frac{1}{2}}}$. For a more interesting stuff, Euler, one of the most celebrated mathematicians of all times, showed that $\zeta(2n) = \frac{(-1)^{n-1}2^{2n-1}B_{2n}\pi^{2n}}{(2n)!}$ (for a proof see for instance [7], pp. $124-125$. It must be noted that Euler $(1707-1783)$ had considered the zeta function only as a real function whereas Riemann $(1826-1866)$ had examined it as a complex function in his masterpiece "On the number of primes less than a given magnitude" where also he stated his famous conjecture that all non-trivial zeros of the zeta function lie on the line $x = \frac{1}{2}$ (Riemann Hypothesis)).
Thus, for example, $\sum_1^\infty \frac{1}{k^2} = \frac{\pi^2}{6}$, $\sum_1^\infty \frac{1}{k^4} = \frac{\pi^4}{90}$, $\sum_1^\infty \frac{1}{k^6} = \frac{\pi^6}{945}$, $\sum_1^\infty \frac{1}{k^{26}} = \frac{1315862}{11094481976030578125}\pi^{26}$. However, Euler was unable to prove any similar results for odd arguments but conjectured [6] that $\zeta(3) := \sum_{k=1}^\infty \frac{1}{k^3} = \alpha(\ln 2)^2 + \beta\frac{\pi^2}{6}\ln 2$, where α and β are rational numbers. For over 300 years the exact value of the convergent series $\sum_{k=1}^\infty \frac{1}{k^3}$ has remained a mystery despite the following interesting results:

- $\zeta(3) = -\frac{5}{6}(\ln\frac{1+\sqrt{5}}{2})^3 + \frac{1}{6}\pi^2\ln\frac{1+\sqrt{5}}{2} + \frac{5}{4}\sum_{n=1}^\infty \frac{1}{n^3(\frac{1+\sqrt{5}}{2})^{2n}}$
  (Landen [11], 1780);
- $\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n=1}^\infty \frac{1}{n^3(e^{2n\pi}-1)}$ (Lerch [12], 1901);
- $\zeta(3) = \frac{5}{2}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3\binom{2n}{n}}$ (R. Apéry [1], 1979)
- $\zeta(3) = \frac{8}{7}\left(\frac{(\ln 2)^3}{3} + \sum_{n=1}^\infty \frac{1}{2^n}(\frac{\ln 2}{n^2} + \frac{1}{n^3})\right)$ (W. Janous [10], 2006)

Remark 1.1. *This author categorized Janous formula as an interesting one for the following reasons:*
Clearly the numerical series $\sum_{n=1}^\infty \frac{1}{2^n}\frac{\ln 2}{n^2}$ *and* $\sum_{n=1}^\infty \frac{1}{2^n}\frac{1}{n^3}$ *are series of positive terms which converge by the comparison test with the series* $\sum_{n=1}^\infty \frac{1}{n^2}$ *and*

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ *respectively. Thus* $\zeta(3) = \frac{8}{7}\frac{(\ln 2)^3}{3} + \frac{8}{7}\ln 2 \sum_{n=1}^{\infty} \frac{1}{2^n n^2} + \frac{8}{7}\sum_{n=1}^{\infty} \frac{1}{2^n n^3}$. *Using the well-known formula* $\sum_{n=1}^{\infty} \frac{1}{2^n n^2} = \frac{\pi^2}{12} - \frac{(\ln 2)^2}{2}$, *we finally get* $\zeta(3) = \frac{-4}{21}(\ln 2)^3 + \frac{2}{21}\pi^2 \ln 2 + \frac{8}{7}\sum_{n=1}^{\infty} \frac{1}{2^n n^3}$ *which in this derived form, to this author's knowledge, is the closest to the above-mentioned Euler's conjecture.*

2 PSLQ in Action

In [3], this author obtained the following result

$$\zeta(3) = -\frac{\sqrt{3}}{18}\pi^3 + \frac{3\sqrt{3}}{4}\pi \sum_{1}^{\infty} \frac{1}{(3n-2)^2} - \frac{3}{4}\sum_{1}^{\infty} \frac{1}{n^3\binom{2n}{n}}.$$

We improve our result by using, as a tool, the most efficient algorithm available at this time which is the PSLQ Integer Relation Algorithm and which is implemented in Maple (The PSLQ Algorithm, due to mathematician Helaman Ferguson, was featured in Science (October 2006) as "the best-known integer relation algorithm" and was named by Computing in Science and Engineering (January 2000) as "one of the top ten algorithms of the 20th century" [2]).

Definitions. Let $r \in \mathbb{R}^n$ be a given vector. We say that the vector $c \in \mathbb{Z}^n$ is an **integer relation** for r if $\sum_{1}^{n} c_k r_k = 0$ with at least one non-zero c_k. An **integer relation algorithm** searches therefore for such a non-zero vector c.

The PSLQ algorithm either finds the integers or obtains lower bounds on the sizes of coefficients for which such a relation is valid. Usually, a high degree of numerical precision is needed for PSLQ to run efficiently. Otherwise, "large" coefficients result suggesting failure in discovering a relation. Let us try to discover the familiar identity $\cos(3x) = 4\cos^3 x - 3\cos x$, to familiarize ourselves with the process (The function $\cos[(2n-1)x]$ can be written as a linear combination of odd powers $\cos x, ..., \cos^{2n-1} x$):

$>$ with(IntegerRelations):Digits:=50; $x := sqrt(3)/2$;

$> PSLQ([cos(3*x), seq(cos(x) \wedge (2*j-1), j = 1..5)])$;

$[1, 3, -4, 0, 0, 0]$

Note that the point of the PSLQ algorithm is to find vectors whose components are small. As a result, we can write $\cos(3x) + 3\cos x - 4\cos^3 x = 0$; that is, $\cos(3x) = 4\cos^3 x - 3\cos x$. Let us confirm our result experimentally

with another value of x:
$>$ with(IntegerRelations):Digits:=50; $x := 0.5$;
$> PSLQ([cos(3*x), seq(cos(x) \wedge (2*j-1), j = 1..5)])$;
$[-1, -3, 4, 0, 0, 0]$

Our idea is now to use the PSLQ algorithm to experimentally find an integer relation among carefully selected series whereas if we suspect a series can be expressed as a rational linear combination of other series, then we evaluate all of these series at some random value. Afterwards, to confirm our result-still experimentally-we use the PSLQ algorithm at another value to produce another set of coefficients. If our second response is a scaled version of the first one, then we conjecture that such a relation has been found. Finally, we prove our conjecture mathematically. Here is an illustration of PSLQ on something we know involving series:
$>$with(IntegerRelations): Digits:=50;

$$Digits := 50$$

$> x := sqrt(3)/2$;

$$x := \frac{\sqrt{3}}{2}$$

$> PSLQ([sum(1/(3*n-2) \wedge 2, n = 1..infinity), seq(cos(x) \wedge (2*j-1), j = 1..5)])$; (That was an intentional random application resulting in large numbers)

[263097680, −436660359, 220554598, −543586674, −183108721, −69263272]

$> PSLQ([sum(1/n \wedge 2, n = 1..infinity), (\ln(2)) \wedge 2, Pi \wedge 2])$; (That was an intentional application resulting in expected small numbers)

$$[-6,\ 0,\ 1]$$

$> PSLQ([sqrt(3)*Pi \wedge 3, sum(1/n \wedge 3, n = 1..infinity), sum(1/(n \wedge 3 * binomial(2*n, n)), n = 1..infinity), sum(sqrt(3)*Pi*1/(3*n-2) \wedge 2, n = 1..infinity)])$;

$$[2,\ 36,\ 27,\ -27]$$

Let us confirm our result experimentally with another value of x:
$>$ with(IntegerRelations):Digits:=50;

$$Digits := 50$$

$> x := 0.45;$

$$x := 0.45$$

$> PSLQ([sqrt(3) * Pi \wedge 3, sum(1/n \wedge 3, n = 1..infinity), sum(1/(n \wedge 3 * binomial(2*n, n)), n = 1..infinity), sum(sqrt(3) * Pi * 1/(3*n-2) \wedge 2, n = 1..infinity)]);$

$$[2,\ 36,\ 27,\ -27]$$

The discovered identity via PSLQ is:

$$2\sqrt{3}\pi^3 + 36\sum_{n=1}^{\infty}\frac{1}{n^3} + 27\sum_{n=1}^{\infty}\frac{1}{n^3\binom{2n}{n}} - 27\sqrt{3}\pi\sum_{n=1}^{\infty}\frac{1}{(3n-2)^2} = 0$$

(a mathematical proof of this was supplied by this author in [8].)

3 New Result

>with(IntegerRelations):Digits:=10; $x := sqrt(3)/2; t := \ln(2)\wedge 2; u := sqrt(2)* Pi; v := sum(1/(n\wedge 3*binomial(2*n, n)), n = 1..infinity); PSLQ([t, u, v, sum(Pi* sqrt(3) * 1/(3*n+1) \wedge 2, n = 0..infinity)]);$

$$Digits := 10$$

$$x := \frac{\sqrt{3}}{2}$$

$$t := \ln(2)^2$$

$$u := \sqrt{2}\,\pi$$

$$v := \frac{1}{2}\,\mathrm{hypergeom}([1,\ 1,\ 1,\ 1],\ [\frac{3}{2},\ 2,\ 2],\ \frac{1}{4})$$

$$[7,\ -9,\ 0,\ 6]$$

Let us confirm our result experimentally with another value of x:
>with(IntegerRelations):Digits:=10;

$$Digits := 10$$

$> x$:=1.342567898;

$$x := 1.342567898$$

$t := \ln(2) \wedge 2; u := sqrt(2) * Pi; v := sum(1/(n \wedge 3 * binomial(2 * n, n)), n = 1..infinity); PSLQ([t, u, v, sum(sqrt(3)*Pi*1/((3*n+1)\wedge 2), n = 0..infinity)]);$

$$t := \ln(2)^2$$

$$u := \sqrt{2}\,\pi$$

$$v := \frac{1}{2}\,\mathrm{hypergeom}([1,\ 1,\ 1,\ 1],\ [\frac{3}{2},\ 2,\ 2],\ \frac{1}{4})$$

$$[7,\ -9,\ 0,\ 6]$$

The discovered identity, using PSLQ, is

$$7\ln^2 2 - 9\sqrt{2}\pi + 6\sqrt{3}\pi \sum_{1}^{\infty} \frac{1}{(3n-2)^2} = 0.$$

4 General Results

Recall that $\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n=1}^{\infty} \frac{1}{n^3(e^{2n\pi}-1)}$. In this section we use PSLQ to try to get results of this type for odd values other than 3 :
Let us start with 5 :
>with(IntegerRelations):Digits:=50;

$$Digits := 50$$

$> x := sqrt(3)/2;$

$$x := \frac{\sqrt{3}}{2}$$

$> PSLQ([sum(1/n \wedge 5, n = 1..infinity), Pi \wedge 5, sum(1/(n \wedge 5 * (exp(2 * Pi * n) - 1)), n = 1..infinity)]);$

$$[239548242096527,\ -804499191833,\ -1176685284797976]$$

$> PSLQ([sum(1/n \wedge 5, n = 1..infinity), Pi \wedge 5, sum(1/(n \wedge 5 * (exp(2 * Pi * n) - 1)), n = 1..infinity), sum(1/(n \wedge 5 * (exp(2 * Pi * n) + 1)), n = 1..infinity)]);$

$$[-1470,\ 5,\ -3024,\ -84]$$

$> x := 1.3456789;$

$$x := 1.3456789$$

$> PSLQ([sum(1/n \wedge 5, n = 1..infinity), Pi \wedge 5, sum(1/(n \wedge 5 * (exp(2 * Pi * n) - 1)), n = 1..infinity), sum(1/(n \wedge 5 * (exp(2 * Pi * n) + 1)), n = 1..infinity)]);$

$$[-1470,\ 5,\ -3024,\ -84]$$

Thus $-1470\sum_{n=1}^{\infty}\frac{1}{n^5} + 5\pi^5 - 3024\sum_{n=1}^{\infty}\frac{1}{n^5(e^{2\pi n}-1)} - 84\sum_{n=1}^{\infty}\frac{1}{n^5(e^{2\pi n}+1)} = 0$. Consequently, $\zeta(5) = \frac{1}{294}\pi^5 - \frac{72}{35}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{2\pi n}-1)} - \frac{2}{35}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{2\pi n}+1)}$.

Now for $\zeta(7)$:

>with(IntegerRelations):Digits:=50;

$$Digits := 50$$

$> x := sqrt(3)/2;$

$$x := \frac{\sqrt{3}}{2}$$

$> PSLQ([sum(1/n \wedge 7, n = 1..infinity), Pi \wedge 7, sum(1/(n \wedge 7 * (exp(2 * Pi * n) - 1)), n = 1..infinity)]);$

$$[-56700,\ 19,\ -113400]$$

$> x := 0.123456;$

$$x := 0.123456$$

$> PSLQ([sum(1/n \wedge 7, n = 1..infinity), Pi \wedge 7, sum(1/(n \wedge 7 * (exp(2 * Pi * n) - 1)), n = 1..infinity)]);$

$$[-56700,\ 19,\ -113400]$$

Thus $-56700\sum_{n=1}^{\infty}\frac{1}{n^7} + 19\pi^7 - 113400\sum_{n=1}^{\infty}\frac{1}{n^5(e^{2\pi n}-1)} = 0$. Consequently, $\zeta(7) = \frac{19}{56700}\pi^7 - 2\sum_{n=1}^{\infty}\frac{1}{n^7(e^{2\pi n}-1)}$. The following is an interesting result of Lerch in 1901 from which our PSLQ-discovered formula for $\zeta(7)$ follows immediately by taking $n = 3$:

Theorem 4.1. *[12] If n is an ODD positive integer, then*

$$\zeta(2n+1) = \frac{(2\pi)^{2n+1}}{2}\sum_{k=0}^{n+1}(-1)^{k+1}\frac{B_{2k}}{(2k)!}\frac{B_{2n+2-2k}}{(2n+2-2k)!}$$

$$-2\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}(e^{2\pi k}-1)}.$$

(This result can be easily phrased as: If n is an ANY positive integer, then

$$\zeta(4n+3) = \frac{(2\pi)^{4n+3}}{2}\sum_{k=0}^{2n+2}(-1)^{k+1}\frac{B_{2k}}{(2k)!}\frac{B_{4n+4-2k}}{(4n+4-2k)!}$$
$$-2\sum_{k=1}^{\infty}\frac{1}{k^{4n+3}(e^{2\pi k}-1)}.)$$

The following result was stated by Ramanujan and later proved by others including Grosswald [9]:

Theorem 4.2. *[13] Let n be a positive integer and α and β are positive numbers with $\alpha\beta = \pi^2$. Then*

$$\frac{1}{\alpha^n}\{\frac{1}{2}\zeta(2n+1)+\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}(e^{2k\alpha}-1)}\} = \frac{(-1)^n}{\beta^n}\{\frac{1}{2}\zeta(2n+1)+\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}(e^{2k\beta}-1)}\}$$
$$+2^{2n}\sum_{k=0}^{n+1}(-1)^{k+1}\frac{B_{2k}}{(2k)!}\frac{B_{2n+2-2k}}{(2n+2-2k)!}\alpha^{n+1-k}\beta^k$$

Remark 4.3. *Lerch's result follows easily from Ramanujan's result by taking $\alpha = \beta = \pi$. However, our PSLQ-discovered formula for $\zeta(5)$ does not follow from Ramanujan's formula as it yields $0 = 0$ for $n = 2$. As Lerch's formula covers the cases $\zeta(3), \zeta(7), \zeta(11), \zeta(15), \zeta(19), ...$; that is, $\zeta(4n+3), n = 0, 1, 2, 3, ...$, our interest is now shifted towards a formula complementing those to cover the remaining cases $\zeta(5), \zeta(9), \zeta(13), \zeta(17), ...$; that is, $\zeta(4n+1), n = 1, 2, 3, ...$ Even though the following interesting two formula were mentioned in [4] and [3] respectively, our PSLQ-discovered formula for $\zeta(5)$ is still obviously not an outcome of them due to the last terms in them:*

$$\zeta(4n+1) = \frac{(2\pi)^{4n+1}}{2}\frac{1}{2n}\sum_{k=0}^{2n+1}(-1)^{k+1}(2k-1)\frac{B_{2k}}{(2k)!}\frac{B_{4n+2-2k}}{(4n+2-2k)!}$$
$$-2\sum_{k=1}^{\infty}\frac{1}{k^{4n+1}(e^{2\pi k}-1)} - \frac{\pi}{2n}\sum_{k=1}^{\infty}\frac{1}{k^{4n}\sinh^2(\pi k)}$$

$$\zeta(4n+1) = \frac{(2\pi)^{4n+1}}{2}\frac{1}{2n}\sum_{k=1}^{2n+1}(-1)^{k+1}\frac{B_{2k}}{(2k-1)!}\frac{B_{4n+2-2k}}{(4n+2-2k)!}$$
$$-\frac{1}{n}\sum_{k=1}^{\infty}\frac{(2\pi k+2n)e^{2\pi k}-2n}{k^{4n+1}(e^{2k\pi}-1)^2}.$$

5 Complementary formula

The result that we are looking for generalizing the PSLQ-discovered formula for $\zeta(5)$ is

$$[1+(-4)^n-2^{4n+1}]\zeta(4n+1)$$

$$= (2\pi)^{4n+1}\sum_{k=0}^{n}(-4)^{k+n}\frac{B_{4k}}{(4k)!}\frac{B_{4n+2-4k}}{(4n+2-4k)!}+\frac{1}{2}(2\pi)^{4n+1}\sum_{k=0}^{2n+1}(-4)^{k}\frac{B_{2k}}{(2k)!}\frac{B_{4n+2-2k}}{(4n+2-2k)!}$$

$$+2[2^{4n+1}-(-4)^n]\sum_{k=1}^{\infty}\frac{1}{k^{4n+1}(e^{2k\pi}-1)}+2\sum_{k=1}^{\infty}\frac{1}{k^{4n+1}(e^{2k\pi}+1)}$$

and is proved in [14].

Remark 5.1. *Even though the preceding formulas go in a direction different from the previous section and do not find the exact values of the zeta function at all odd values, they seem to be the best available general results so far.*

Acknowledgment. This research was partially supported financially by the National Council for Scientific Research-Lebanon.

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