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Effective construction of irreducible curve singularities

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Abstract

By using the effective notion of the approximate roots of a polynomial, we describe the equisingularity classes of irreducible curve singularities with a given Milnor number.

Introduction

Let $\mathbf{K}$ be an algebraically closed field of characteristic zero. Let f be an irreducible monic polynomial of $\mathbf{R} = \mathbf{K}[[x]][y]$, say $f = f(x, y) = y^n + a_1(x).y^{n-1} + \ldots + a_n(x) \in \mathbf{R}$. Up to a change of coordinates, we assume that $a_1(x) = 0$. For all $g \in \mathbf{R}$ let $\mathrm{int}(f, g)$ denote the intersection multiplicity of f and g. Let $\Gamma(f) = \{\mathrm{int}(f, g) : g \in \mathbf{R} - (f)\}$ be the semigroup of f. If f' is another irreducible polynomial of $\mathbf{R}$, then f and f' are said to be *equisingular* if $\Gamma(f) = \Gamma(f')$ (for example, $y^2 - x^3$ and $y^3 - x^2$ are equisingular because they are both associated with the semigroup generated by $2, 3$. In particular, two equisingular polynomials of $\mathbf{R}$ need not have the same degree in y). It is well-known that in this case $\mu(f) = \mu(f')$, where $\mu(f) = int(f_x, f_y)$ is called the *Milnor number* of f. The converse is false. The *equisingularity class* of the polynomial f is the set of irreducible polynomials of $\mathbf{R}$ which are equisingular to f. It is of a certain interest to determine this equisingularity class, which gives a classification of the polynomials of $\mathbf{R}$ in terms of subsemigroups of $\mathbf{Z}$. Another remarkable classification is obtained if one can characterize all polynomials whose Milnor number is equal to some fixed nonnegative integer m. The aim of this paper is to study the two questions from an effective point of view: we first give, for a fixed semigroup of an irreducible polynomial f of $\mathbf{R}$, all elements of the equisingularity class of f. Then, for a fixed m in $\mathbf{N}$, by similar methods we construct the generic forms of all irreducible polynomials f of $\mathbf{R}$ such that $\mu(f) = m$. The set of these polynomials is the union of a finite number of equisingularity classes. We think that this effective classification is useful in the study of problems and conjectures in the theory of irreducible curve singularities, particularly in the

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understanding of their moduli spaces. Our approach uses the effective notion of approximate roots of a polynomial f of $\mathbf{R}$ introduced by S.S. Abhyankar and T.T. Moh and the notion of generalized Newton polygon introduced by Abhyankar. The first one gives rise to an algorithm for the computation of the set of generators of the semigroup of f (and then the set of Newton-Puiseux pairs of f, see Definition 1.3, and [1], [2]). The second one is used by Abhyankar to give an irreducibility criterion for the polynomial f (see [3]). We would like to point out that our algorithms are intrinsic and that they have been implemented with *Mathematica* (see [8]), and *Maple*.

1 Characteristic sequences

In this Section we recall the notion of approximate roots of f as well as the characteristic sequences associated with an irreducible polynomial $f = y^n + a_2(x).y^{n-2} + \ldots + a_n(x)$ of $\mathbf{R}$.

Definition 1.1 For any monic polynomial $g \in \mathbf{R}$, the **intersection multiplicity** $\text{int}(f,g)$ of f with g is the x-order of the y-resultant of f and g.

The set $\Gamma(f) = \{\text{int}(f,g) : g \in \mathbf{R} - f\}$ is a subsemigroup of $\mathbf{Z}$, called the **semigroup of** f.

Definition 1.2 Let $y(t) = \sum_j a_j t^j \in \mathbf{K}[[t]]$ be a root of $f(t^n, y) = 0$, according to Newton Theorem. Set $m_0 = d_1 = n, m_1 = \inf\{j : a_j \neq 0\}$, and for all $k \geq 1$, let
$m_{k+1} = \inf\{j : a_j \neq 0 \text{ and } d_k \text{ does not divide } j\}$, and $d_{k+1} = \gcd(m_{k+1}, d_k)$.
Since f is irreducible, there exists h such that $d_{h+1} = 1$. Set $m_{h+1} = \infty$.
Finally, set $r_0 = m_0 = n$, $r_1 = O_x(a_n(x))$, where O_x denotes the x-order, and for all $k = 1, \ldots, h-1$:

$$r_{k+1} = r_k(\frac{d_k}{d_{k+1}}) + (m_{k+1} - m_k).$$

(Note that, since $a_1(x) = 0$, we have $r_1 = m_1$).
We recall that $r_0, \ldots, r_h$ generates the semigroup $\Gamma(f)$ of f. We use the notation $\Gamma(f) =< r_0, \ldots, r_h >$.

Definition 1.3 For all $k = 1, \ldots, h$, set $e_k = \dfrac{d_k}{d_{k+1}}$. The set $\{(\dfrac{m_k}{d_{k+1}}, e_k) : 1 \leq k \leq h\}$ is called the set of **Newton-Puiseux pairs** of f.

Definition 1.4 Let d be a positive integer and assume that d divides n. Let g be a monic polynomial of $\mathbf{R}$, of degree $\dfrac{n}{d}$ in y. We call g the d-th approximate root of f if one of the following holds:

i) $\deg_y(f - g^d) < n - \dfrac{n}{d}$.

ii) in the expansion $f = g^d + \alpha_1 g^{d-1} + \ldots + \alpha_d$ of f with respect to the powers of g, $\alpha_1 = 0$.

We note that i) and ii) are equivalent.
We denote the d-th approximate root of f by $\mathrm{App}_d(f)$. It is clear that $\mathrm{App}_d(f)$ is unique, and also that it is effectively computable if the series $a_k(x), k = 2, \ldots, n$, are polynomials.

Remark 1.5 Given a divisor d of n, the d-th approximate root $\mathrm{App}_d(f)$ of f can be effectively constructed from the equation of f in the following way:
Take $G_0 = y^{n/d}$ and let $f = G_0^d + \alpha_1^0 G_0^{d-1} + \ldots + \alpha_d^0$ be the expansion of f with respect to the powers of G_0.

i) If $\alpha_1^0 = 0$, then $G_0 = \mathrm{App}_d(f)$.

ii) If $\alpha_1^0 \neq 0$, then set $G_1 = G_0 + \dfrac{\alpha_1^0}{d}$ and consider the expansion $f = G_1^d + \alpha_1^1 G_1^{d-1} + \ldots + \alpha_d^1$ of f with respect to the powers of G_1. If $\alpha_1^1 \neq 0$, then easy calculations show that $\deg_y \alpha_1^0 > \deg_y \alpha_1^1$. This process stops after a finite number of steps, constructing $\mathrm{App}_d(f)$.

Remark 1.6 If the characteristic of $\mathbf{K}$ is not zero and if this characteristic does not divide n, then the construction above applies without any restriction. Otherwise, the theory of approximate roots does not work as it is. More information can be found in [9].

Let $g_1, \ldots, g_h, g_{h+1}$ be the d_k-th approximate roots of f for $k = 1, \ldots, h+1$ (in particular, $g_1 = y$ and $g_{h+1} = f$).

Lemma 1.7 *(see [1], (8.2) The Fundamental Theorem (part one)) For all $k = 1, \ldots, h$, we have:*

i) $\mathrm{int}(f, g_k) = r_k$.

ii) g_k is irreducible in $\mathbf{R}$ *and* $\Gamma(g_k) =< \dfrac{r_0}{d_k}, \ldots, \dfrac{r_{k-1}}{d_k} >$. *Furthermore, $g_1, \ldots, g_{k-1}$ are the approximate roots of g_k.*

Lemma 1.8 *(see [13]) The following formulas hold:*

$$\mathrm{int}(f_x, f_y) = \sum_{i=1}^{h} (e_i - 1) r_i - n + 1. \text{ In particular, } \mathrm{int}(f_x, f_y) \text{ is even.}$$

$$\text{For all } k = 2, \ldots, h,\ \mathrm{int}(f_x, f_y) = d_k.\mathrm{int}(g_{k_x}, g_{k_y}) + \sum_{i=k}^{h} (e_i - 1) r_i - d_k + 1.$$

Proof. The proof of the first formula can be found in [13] (3.14, p. 18). The second formula results from the first one by easy calculations.

Remark 1.9 The intersection multiplicity $\mathrm{int}(f_x, f_y)$ is also called the Milnor number of f. It is an invariant of f and, by the formula above, it is common to the elements of the equisingularity class of f. It also coincides with the conductor of the semigroup $\Gamma(f)$, usually denoted by c, and has the following numerical characterization: for all $p \geq c, p \in \Gamma(f)$. Furthermore, given two integers a, b, if $a + b = m - 1$, then exactly one of $a, b \in \Gamma(f)$. Since $\Gamma(f)$ has no negative integers, it follows that $\text{Card}(\mathbf{N} - \Gamma) = \frac{m}{2}$. In fact, c is nothing but the order of the conductor of the quotient $\frac{\mathbf{R}}{(f)}$ into its integral closure. Unlike the Milnor number, the conductor can be defined without restriction on the characteristic of $\mathbf{K}$. An exhaustive exposition of this theory in positive characteristic can be found in [9].

2 Generalized Newton polygons and the irreducibility criterion of Abhyankar

Let $f = y^n + a_2(x)y^{n-2} + \ldots + a_n(x)$ be a monic polynomial, not necessarily irreducible in $\mathbf{R}$. In this section, the notation introduced above will have a more general meaning: $r = (r_0 = n, r_1, \ldots, r_h)$ will denote any sequence of integers such that $r_k < r_{k+1}$ for all $k = 1, \ldots, h-1$, and we shall set $d_{k+1} = \gcd(r_0, r_1, \ldots, r_k)$ for all $k = 0, \ldots, h$. For all $k = 1, \ldots, h$, we set $e_k = \frac{d_k}{d_{k+1}}$; $g = (g_1, \ldots, g_h, g_{h+1} = f)$ will be a sequence of monic polynomials of $\mathbf{R}$ such that $\deg_y g_k = \frac{n}{d_k}$ for all $k = 1, \ldots, h$. We recall some important properties.

Theorem 2.1 (see [1], (8.3) The Fundamental Theorem (part two)) Let

$$B = \{b = (b_1, b_2, \ldots, b_h, b_{h+1}) \in \mathbf{N}^{h+1} : b_1 < e_1, \ldots, b_h < e_h, b_{h+1} < \infty\}.$$

For all $b \in B$, denote $g^b = g_1^{b_1}. \ldots .g_h^{b_h}.f^{b_{h+1}}$, then we have:

i) $\mathbf{R} = \sum_{b \in B} \mathbf{K}[[x]].g^b$.

ii) Let p be a polynomial of $\mathbf{R}$ and write $p = \sum_{k=1}^{s} a_k(x).g^{b^k}$, where $b^k \in B$ for all $k = 1, \ldots, s$. Moreover, let $b_0^k = \mathrm{O}_x a_k(x)$, then associate with any "monomial" $a_k(x).g^{b^k}$ the integer $< (b_0^k, b_1^k, \ldots, b_h^k), r >= b_0^k.r_0 + \sum_{i=1}^{h} b_i^k.r_i$. Finally, let $B' = \{b^k; b_{h+1}^k = 0\}$. With this notation we have:

1) If B' contains at least two elements, then for all $b^i, b^j \in B'$,

$$b^i \neq b^j \Longleftrightarrow < (b_0^i, b_1^i, \ldots, b_h^i), r > \neq < (b_0^j, b_1^j, \ldots, b_h^j), r >$$

2) f does not divide p iff $B' \neq \emptyset$, and in this case there is a unique k_0 such that $< (b_0^{k_0}, b_1^{k_0}, \ldots, b_h^{k_0}), r >= \inf\{< (b_0^k, b_1^k, \ldots, b_h^h), r >; b^k \in B'\}$.

Definition 2.2 (see [3]) The integer $< (b_0^{k_0}, b_1^{k_0}, \ldots, b_h^{k_0}), r >$ is called the **formal intersection multiplicity** of p with respect to (r, g) and will be denoted by $\text{fint}(p, r, g)$.

Now we recall the notion of **generalized Newton polygon**. Let p be a monic polynomial of $\mathbf{R}$ of degree n in y and consider a monic polynomial q of $\mathbf{R}$ of degree $\frac{n}{d}$ in y, where d is a divisor of n. Let

$$p = q^d + \alpha_1(x,y)q^{d-1} + \ldots + \alpha_d(x,y)$$

be the expansion of p with respect to the powers of q, and consider the sequences r, g defined above. One associates with p the generalized Newton polygon which is defined as the union of all compact sides of the convex hull in $\mathbf{R^2}$ of the set formed by the points $(\text{fint}(\alpha_k, r, g), (d-k).\text{fint}(q, r, g))$ for all $1 \leq k \leq d$. It will be denoted by $\text{GNP}(p, q, r, g)$ (see [3]).
With this notation we have:

Irreducibility criterion (see [3])

Write $p = y^n + a_1(x)y^{n-1} + \ldots + a_n(x) \in \mathbf{R}$ and assume that, possibly after a change of variables, $a_1(x) = 0$. Consider the sequences r_k, g_k, d_k defined by:

$r_0 = d_1 = n$

$g_1 = y, r_1 = \text{int}(p, g_1)$, and for all $k \geq 2$:

$d_k = \text{gcd}\ (r_0, r_1, \ldots, r_{k-1}), g_k = \text{App}_{d_k}(p), r_k = \text{int}(p, g_k)$.

p is irreducible if and only if the following conditions hold:

1) There is $h \in N$ such that $d_{h+1} = 1$.

2) For all $k = 1, \ldots, h-1$, $r_{k+1}d_{k+1} > r_k d_k$.

3) Set $p = g_{h+1}$ and for all $k = 1, \ldots, h$, let $e_k = \frac{d_k}{d_{k+1}}$ and $g^k = (g_1, \ldots, g_k)$.

For all $k = 1, \ldots, h$, let $\frac{r}{d_{k+1}} = (\frac{r_0}{d_{k+1}}, \ldots, \frac{r_k}{d_{k+1}})$. Then for all $k = 1, \ldots, h$, the generalized Newton polygon $\text{GNP}(g_{k+1}, g_k, \frac{r}{d_{k+1}}, g^k)$ is the line segment joining $(0, \frac{r_k}{d_{k+1}}.e_k)$ and $(\frac{r_k}{d_{k+1}}.e_k, 0)$.

Remarks 2.3 i) (see [3]) Suppose that p is irreducible, and let $r = (r_0, r_1, \ldots, r_h)$ and $g = (g_1 = y, g_2, \ldots, g_h, g_{h+1} = p)$ be the sequences defined above. Let p' by a polynomial of $\mathbf{R}$ and consider the expansion of p' with respect to the sequences r, g (see Theorem 2.1). If the corresponding set B' is not empty, then $\text{fint}(p', r, g) = \text{int}(p', p)$.

ii) Part 3) of the criterion can be made more precise as follows: whenever p is irreducible, the generalized Newton polygon $\text{GNP}(g_{k+1}, g_k, \frac{r}{d_{k+1}}, g^k)$ just contains the two extremal points for all $k = 1, \ldots, h-1$. In fact, let $k \in \{1, \ldots, h\}$ and let $g_{k+1} = g_k^{e_k} + \alpha_2(x,y).g_k^{e_k-2} + \ldots + \alpha_{e_k}(x,y)$ be the expansion of g_{k+1} w.r.to g_k, then we have:

a) $\text{int}(g_{k+1}, \alpha_{e_k}(x,y)) = \frac{r_k}{d_{k+1}}.e_k = \text{int}(g_{k+1}, g_k^{e_k})$.

b) For all $i = 2, \ldots, e_k - 1$, $\text{int}(g_{k+1}, \alpha_i(x,y)) > \frac{r_k}{d_{k+1}}.i$, In particular, $\text{int}(g_{k+1}, \alpha_i(x,y)) + \text{int}(g_{k+1}, g_k^{e_k-i}) > \frac{r_k}{d_{k+1}}.e_k$.

iii) (see [13]) As an immediate consequence of ii) a) we have, for all $k = 1, \dots, h$,

$$\frac{r_k}{d_{k+1}}.e_k \in e_k.\Gamma(g_k) =< \frac{r_0}{d_{k+1}}, \dots, \frac{r_{k-1}}{d_{k+1}} > .$$

In particular,

$$r_k.e_k \in < r_0, \dots, r_{k-1} > .$$

3 Constructing the equisingularity class

In this Section, fix a semigroup of nonnegative integers $\Gamma =< r_0, r_1, \dots, r_h >$ and set $d_1 = r_0$ and $d_{k+1} = \gcd(r_0, \dots, r_k)$ for all $k = 1, \dots, h$ (by convention $r_{h+1} = d_{h+2} = \infty$). Moreover, assume $d_{k+1} = 1$ and $r_{k+1}.d_{k+1} > r_k.d_k$ for all $k = 1, \dots, h$ (*) (this condition appeared in the irreducibility criterion of Section 2). This implies that the sequence $(r_1, \dots, r_h)$ is strictly increasing. This also holds if r_1 is replaced by r_0.

Let $\tilde{\mathbf{R}}$ denote the set of all irreducible monic polynomials f of $\mathbf{R}$ of the form $f = f(x, y) = y^n + a_2(x).y^{n-2} + \dots + a_n(x)$. Condition $(*)$ implies that there exists a polynomial $f \in \tilde{\mathbf{R}}$ such that $\Gamma = \Gamma(f)$ (see [13]). Here we give the generic forms of all these polynomials; i.e., we describe the set of elements of $\tilde{\mathbf{R}}$ having the semigroup Γ. The construction can be done with respect to the arrangements $(r_0, r_1, r_2, \dots, r_h)$ and $(r_1, r_0, r_2, \dots, r_h)$ and we choose to do it with respect to the first arrangement(The polynomials that we would get with respect to the second arrangement are those obtained by exchanging x and y.)

Throughout this paper we assume that $r_0, \dots, r_h$ form a minimal system of generators for Γ. This condition can be reformulated equivalently as a numerical criterion. First, we derive a useful identity:
For all $1 \leq i \leq h, sete_i = \dfrac{d_i}{d_{i+1}}$. For all $2 \leq k \leq h$ we have:

$$\begin{aligned} \sum_{i=1}^{k-1}(e_i - 1)r_i &= \sum_{i=1}^{k-1}(r_{i+1} - (m_{i+1} - m_i)) - \sum_{i=1}^{k-1} r_i \\ &= r_k - r_1 - \sum_{i=1}^{k-1}(m_{i+1} - m_i) = r_k - m_k \quad (**) \end{aligned}$$

Now, we can prove the following

Lemma 3.1 *Suppose that $r_0, \dots, r_h$ satisfy condition $(*)$. These numbers form a minimal system of generators for the semigroup Γ if and only if $d_1 > d_2 > \dots > d_h > d_{h+1}$.*

Proof. Notice that $d_k \geq d_{k+1}$ for all $k = 1,\ldots,h$. Moreover, recall that the minimality of the system of generators is equivalent to the condition that $r_k \notin < r_0,\ldots,r_{k-1} >$ for all $k = 1,\ldots,h$. First, suppose that this condition is not satisfied for some index k. Then $d_k = \gcd(r_0,\ldots,r_{k-1})$ divides r_k. Hence $d_{k+1} = \gcd(d_k, r_k) = d_k$. For the converse, fix an index k and suppose that $d_{k+1} = d_k$. Then there are integers $\alpha_0,\ldots,\alpha_{k-1}$ such that

$$r_k = \sum_{i=0}^{k-1} \alpha_i r_i. \qquad (1)$$

Now let β_{k-1} be the (nonnegative) remainder of the Euclidean division of α_{k-1} by $e_{k-1} = \dfrac{d_{k-1}}{d_k}$. Since the semigroup Γ satisfies condition (*), Γ is the semigroup of a polynomial of $\tilde{\mathbf{R}}$. In particular, by Remark 2.3 iii), $r_{k-1}.(e_{k-1}) \in < r_0,\ldots,r_{k-2} >$. Hence we can transform (1) in such a way that $\alpha_{k-1} = \beta_{k-1}$. If we successively perform the same procedure for the indices $k-2,\ldots,1$, we finally obtain that in (1) $0 \leq \alpha_i < e_i$ for all $i = 1,\ldots,k-1$. Now by (**)

$$\alpha_0 r_0 = r_k - \sum_{i=1}^{k-1} \alpha_i r_i > r_k - \sum_{i=1}^{k-1} (e_i - 1) r_i = m_k > 0,$$

hence $\alpha_0 > 0$. This proves that $r_k \in < r_0,\ldots,r_{k-1} >$ and completes the proof.
The construction of the generic form of all polynomials $f \in \tilde{\mathbf{R}}$ having Γ as a semigroup is based on the notion of **generalized Newton polygons** introduced in Section 2. We shall recursively construct the sequence of approximate roots $g_1,\ldots,g_h,g_{h+1} = f$.

Let $g_1 = \mathrm{App}_{d_1}(f)$ (recall that, since $a_1(x) = 0$, $g_1 = y$). From Section 2 we know that $g_2 = \mathrm{App}_{d_2}(f)$ satisfies:

i) $\Gamma(g_2) = < \dfrac{r_0}{d_2}, \dfrac{r_1}{d_2} >$.

ii) $\mathrm{GNP}(g_2, g_1, \dfrac{r}{d_2}, g^1)$ is the line segment joining the two points $(0, \dfrac{r_1}{d_2}.e_1)$ and $(\dfrac{r_1}{d_2}.e_1, 0)$.

By virtue of part ii) of Remarks 2.3:

$$g_2 = y^{\frac{r_0}{d_2}} + a.x^{\frac{r_1}{d_2}} + \sum_{i.\frac{r_0}{d_2}+j.\frac{r_1}{d_2} > \frac{n}{d_2}.\frac{r_1}{d_2};\, 0 \leq j < \frac{d_1}{d_2} = \frac{r_0}{d_2}} a_{ij} x^i y^j,$$

where $a \in \mathbf{K} - 0$ and for all (i,j), $a_{ij} \in \mathbf{K}$.
Suppose that we have the generic forms of $g_1,\ldots,g_k$ and consider the expansion of g_{k+1} with respect to g_k:

$g_{k+1} = g_k^{e_k} + \alpha_2(x,y) g_k^{e_k - 2} + \ldots + \alpha_{e_k}(x,y)$.

From Section 2 we know that:

i) $\Gamma(g_{k+1}) = < \dfrac{r_0}{d_{k+1}},\ldots,\dfrac{r_k}{d_{k+1}} >$.

ii) $\mathrm{GNP}(g_{k+1}, g_k, \frac{r}{d_{k+1}}, g^k)$ is the line segment joining the two points $(0, \frac{r_k}{d_{k+1}}.e_k)$ and $(\frac{r_k}{d_{k+1}}.e_k, 0)$.
It follows from Remarks 2.3 that

(1) $\mathrm{int}(g_{k+1}, \alpha_{e_k}(x,y)) = \mathrm{fint}(\alpha_{e_k}(x,y), \frac{r}{d_{k+1}}, g^k) = \frac{r_k}{d_{k+1}}.e_k$,

and that for all $i = 2, \ldots, e_k - 1$:

(2) $\mathrm{int}(g_{k+1}, \alpha_i(x,y)) = \mathrm{fint}(\alpha_i(x,y), \frac{r}{d_{k+1}}, g^k) > \frac{r_k}{d_{k+1}}.i$.

Let B^k be the set of all $\theta = (\theta_0, \ldots, \theta_{k-1}) \in \mathbf{N}^k$ such that $0 \leq \theta_j < e_j$ for all $j = 1 \ldots, k-1$. With each $\theta \in B^k$, associate the "monomial" $M_\theta = x^{\theta_0}.g_1^{\theta_1}.\ldots.g_{k-1}^{\theta_{k-1}}$. For all $i \in \mathbf{N}$ and for all $\theta \in B^k$, we say that M_θ is of type $(k,i,1)$ (respectively of type $(k,i,2)$) if

$$\frac{r_k}{d_{k+1}}.i = \theta_0.\frac{r_0}{d_{k+1}} + \theta_1.\frac{r_1}{d_{k+1}} + \ldots + \theta_{k-1}.\frac{r_{k-1}}{d_{k+1}}$$

respectively

$$\frac{r_k}{d_{k+1}}.i < \theta_0.\frac{r_0}{d_{k+1}} + \theta_1.\frac{r_1}{d_{k+1}} + \ldots + \theta_{k-1}.\frac{r_{k-1}}{d_{k+1}}).$$

Let $E(k,i,1)$ (respectively $E(k,i,2)$) be the set of monomials $M_\theta, \theta \in B^k$, of type $(k,i,1)$ (respectively of type $(k,i,2)$). Since $\frac{r_k}{d_{k+1}}.e_k \in < \frac{r_0}{d_{k+1}}, \ldots, \frac{r_{k-1}}{d_{k+1}} >$, $E(k, e_k, 1)$ is reduced to one element. If we write this element as $M_{\theta^k} = x^{\theta_0}.g_1^{\theta_1}.\ldots.g_{k-1}^{\theta_{k-1}}$, then $(\theta_0, \theta_1, \ldots, \theta_{k-1})$ can be calculated by Euclidean division.
Using Remarks 2.3, (1) and (2) lead to the following generic forms of $\alpha_2, \ldots, \alpha_{e_k}$:

$$\alpha_{e_k} = a.M_{\theta^k} + \sum_{M_\theta \in E(k,e_k,2)} a_\theta.M_\theta,$$

respectively for all $i = 2, \ldots, e_k - 1$,

$$\alpha_i = \sum_{M_\theta^i \in E(k,i,2)} a_\theta^i.M_\theta^i),$$

where $a \in \mathbf{K} - 0$, and for all $\theta, a_\theta \in \mathbf{K}$ (respectively for all θ and for all $i = 2, \ldots, e_k - 1$, $a_\theta^i \in \mathbf{K}$).

Remark 3.2 We proved that, if Γ is the semigroup of a polynomial $f \in \tilde{\mathbf{R}}$, then f and its approximate roots $g_1, \ldots, g_h$ belong to the set of polynomials constructed above. Conversely, let $(g_1, \ldots, g_h, g_{h+1} = f)$ be as above, then the "only if" part of the irreducibility criterion of Abhyankar shows that f is irreducible.

Remark 3.3 Given $1 \leq k \leq h$, it follows from the above construction that a polynomial g_{k+1} may have an infinite number of monomials. In particular, the above construction is not algorithmic. Note, however, that g_{k+1} is obtained from the sum $g_k^{e_k} + a.M_{\theta^k}, a \in \mathbf{K} - 0$ by adding monomials satisfying some conditions. This suggests the introduction of the following set of polynomials: let $G_1 = y$ and for all $1 \le k \le h$, $G_{k+1} = G_k^{e_k} - M_{\theta^k}$. For all $1 \le k \le h$, g_k is obtained from G_k in an obvious way. Set $G = (G_1, \ldots, G_h, G_{h+1})$ and call it the canonical element of the set of all $(g_1, \ldots, g_h, g_{h+1})$ constructed above. The above calculation leads to an algorithm that computes this canonical element. It is based on Euclidean division in $\mathbf{N}$. The different steps can be summarized as follows:

i) Consider a sequence of integers $r_0 < \ldots < r_h$.

ii) Compute the gcd sequence $d = (d_1, \ldots, d_h, d_{h+1})$ such that $d_1 = r_0$ and for all $2 \le k \le h+1, d_k = \gcd(r_{k-1}, d_{k-1})$. Let $e_k = \dfrac{d_k}{d_{k+1}}$ for all $1 \le k \le h$.

iii) If either $d_{h+1} > 1$, or $r_k \cdot e_k \ge r_{k+1}$ for at least one $k, 1 \le k \le h$, then the sequence $(r_0, \ldots, r_h)$ is not the semigroup of an irreducible polynomial of $\tilde{\mathbf{R}}$.

iv) Assume that $d_{h+1} = 1$ and that $r_k \cdot d_k < r_{k+1} \cdot d_{k+1}$ for all $1 \le k \le h-1$.

a) If $d_k = d_{k+1}$ for some $1 \le k \le h$, then eliminate r_k from the r-sequence of i).

b) Assume that $d_1 > d_2 > \ldots > d_h > d_{h+1} = 1$. Then for all $1 \le k \le h$, compute (the unique) $\theta^k = (\theta_0^k, \ldots, \theta_{k-1}^k)$ such that $0 \le \theta_j^k < e_j$ for all $j = 1 \ldots, k-1$ and $\dfrac{r_k}{d_{k+1}}.e_k = \theta_0^k.\dfrac{r_0}{d_{k+1}} + \theta_1^k.\dfrac{r_1}{d_{k+1}} + \ldots + \theta_{k-1}^k \cdot \dfrac{r_{k-1}}{d_{k+1}}$.

c) The canonical element is $G = (G_1, \ldots, G_h, G_{h+1})$ where $G_1 = y$ and for all $2 \le k \le h+1$, $G_k = G_{k-1}^{e_{k-1}} - x^{\theta_0^k} y^{\theta_1^k} \cdot \ldots \cdot G_{k-1}^{\theta_{k-1}^k}$.

This algorithm has been implemented with *Mathematica* (see [8]), and *Maple*: the input is an increasing sequence of positive integers. Then the output is "false" if this sequence does not generate the semigroup of an irreducible polynomial of $\tilde{\mathbf{R}}$. Otherwise, we get the canonical element described above.

Note that our implementation is based on the following: given $r_0, r_1, \ldots, r_{k-1}$, we need to compute the unique $\theta^k = (\theta_0^k, \ldots, \theta_{k-1}^k)$ such that $0 \le \theta_j^k < e_j$ for all $j = 1 \ldots, k-1$ and $\dfrac{r_k}{d_{k+1}}.e_k = \theta_0^k.\dfrac{r_0}{d_{k+1}} + \theta_1^k.\dfrac{r_1}{d_{k+1}} + \cdots + \theta_{k-1}^k.\dfrac{r_{k-1}}{d_{k+1}}$. Instead of applying the Euclidean division, we preferred to scan lists of values, namely the set of values $(a_0, a_1, \ldots, a_{k-1})$ where for all $i \ge 1, 0 \le a_i < e_i$ and $0 \le a_0 \le \dfrac{r_k}{d_{k+1}}.e_k.\dfrac{d_{k+1}}{r_0} = \dfrac{r_k.e_k}{r_0}$. The cardinality of this set is:

$$\frac{r_k.e_k}{r_0}.\prod_{i=1}^{k-1} e_k = \frac{r_k.e_k}{r_0}.\frac{d_1}{d_k} = \frac{r_k}{d_{k+1}}$$

In conclusion, the set of the values scanned in the algorithm is bounded by $\displaystyle\sum_{k=1}^{h} \frac{r_k}{d_{k+1}}$.

Remark 3.4 An element f whose semigroup is Γ can also be calculated by using the theory of Gröbner bases: a reduced Gröbner basis with respect to any well-ordering on $\mathbf{N}^3$ that eliminates t from the equations $x-t^n, y-t^{m_1}-\ldots-t^{m_r}$ contains a unique polynomial $f(x,y)$. If we consider f as an element of $\mathbf{K}[[x,y]]$, then obviously $\Gamma =< r_0,\ldots,r_h >$ is the semigroup of f. It is well known that the complexity of a Gröbner basis is in general doubly exponential. Moreover, the algorithm computes more than we need. We think that our option is more natural in view of our situation, especially because of its complexity and because the output is expressed in terms of the polynomial f.

Example Let $\Gamma =< 8, 12, 50, 101 >$. Here $h = 3$, the r-sequence is $r = (8, 12, 50, 101)$, and the gcd-sequence is $d = (8, 4, 2, 1)$. Moreover, $e_1 = e_2 = e_3 = 2$. Let us construct the canonical element $G = (G_1, G_2, G_3, G_4)$ following the algorithm above. Here we start directly from point iv), b):

$k = 1 : \dfrac{r_1}{d_2} \cdot e_1 = 3 \cdot 2 = \theta_0^1 \cdot \dfrac{r_0}{d_2} = \theta_0^1 \cdot 2$ implies that $\theta_0^1 = 3$.

$k = 2 : 50 = \dfrac{r_2}{d_3} \cdot e_2 = \theta_0^2 \cdot \dfrac{r_0}{d_3} + \theta_1^2 \cdot \dfrac{r_1}{d_3} = \theta_0^2 \cdot 4 + \theta_1^2 \cdot 6$ with $0 \leq \theta_1^2 < 2$. This implies that $\theta_1^2 = 1$, and $\theta_0^2 = 11$.

$k = 3 : 202 = \dfrac{r_3}{d_4} \cdot e_3 = \theta_0^3 \cdot \dfrac{r_0}{d_4} + \theta_1^3 \cdot \dfrac{r_1}{d_4} + \theta_2^3 \cdot \dfrac{r_2}{d_4} = \theta_0^3 \cdot 8 + \theta_1^3 \cdot 12 + \theta_2^3 \cdot 50$ with $0 \leq \theta_1^3, \theta_2^3 < 2$. This implies that $\theta_2^3 = 1, \theta_1^3 = 0, \theta_0^3 = 19$.

In particular, $G_1 = y, G_2 = G_1^2 - x^3 = y^2 - x^3, G_3 = G_2^2 - x^{11} \cdot G_1 = (y^2 - x^3)^2 - x^{11} \cdot y, G_4 = G_3^2 - x^{19} \cdot G_2 = [(y^2 - x^3)^2 - x^{11} \cdot y]^2 - x^{19} \cdot (y^2 - x^3)$.

With the same notation as above, the set of elements $(g_1, g_2, g_3, g_4 = f)$ is then given by:

$$g_1 = y.$$

$$g_2 = y^2 + \alpha_2(x) = y^2 + ax^3 + \sum_{M_\theta \in E(1,2,2)} a_\theta.M_\theta,$$

where $a \in \mathbf{K} - 0$, and for all θ, one has $a_\theta \in \mathbf{K}$ and $M_\theta = x^{\theta_0}$, with $6 < 2\theta_0$. Moreover, $g_3 = g_2^2 + \alpha_2'(x,y) = g_2^2 + a'x^{11}y + \sum_{M_\theta' \in E(2,2,2)} a_\theta'.M_\theta'$, where $a' \in \mathbf{K} - 0$, and for all θ, $a_\theta' \in \mathbf{K}$; for all θ, $M_\theta' = x^{\theta_0'} y^{\theta_1'}$, with $50 < 4\theta_0' + 6\theta_1'$. Finally, $f = g_3^2 + \alpha_2''(x,y) = g_3^2 + a''x^{19}g_2 + \sum_{M_\theta'' \in E(3,2,2)} a_\theta''.M_\theta''$, where $a'' \in \mathbf{K} - 0$, and for all θ, $a_\theta'' \in \mathbf{K}$; for all θ, $M_\theta'' = x^{\theta_0} y^{\theta_1} g_2^{\theta_2}$, with $202 < 8\theta_0'' + 12\theta_1'' + 50\theta_2''$. Hence the generic form of all polynomials having Γ as a semigroup is $f = [(y^2 + ax^3 + F)^2 + a'x^{11}y + F']^2 + a''x^{19}(y^2 + ax^3 + F) + F''$, where $a, a', a'' \in \mathbf{K} - 0$ and F, F' and F'' are arbitrary linear combinations of monomials from $E(1, 2, 2)$, $E(2, 2, 2)$ and $E(3, 2, 2)$ respectively.

Remark 3.5 i) The construction above does not depend on the choice of the coefficients in the field $\mathbf{K}$, provided that it is of characteristic zero; in particular, the algorithm described allows us to work over any subring A of $\mathbf{K}$. If $A = k[t_1,\ldots,t_m]$ is a polynomial ring over a field k of characteristic zero and $\mathbf{K}$ is the algebraic closure of A in its fractions field, then we get the equisingularity class of the $(t_1,\ldots,t_m)$-generic section.

ii) The restriction to the zero characteristic is made only because of the use of the approximate roots in the algorithm. If the characteristic of $\mathbf{K}$ does not divide r_0, then everything above applies (see Remark 1.6). Note that a more general irreducibility criterion has been given by A. Granja (see [12]), but it does not seem to be in computational form.

4 Equisingularity classes with a given Milnor number

In this Section we generalize the results of Section 3: Let $m \in \mathbf{N}$ be a fixed integer. If $m \in 2.\mathbf{N}$, then there exists a polynomial $f = y^n + a_2(x).y^{n-2} + \ldots + a_n(x) \in \tilde{\mathbf{R}}$ such that $\text{int}(f_x, f_y) = m$. Here we shall give the generic forms of all these polynomials. Note that if g is another polynomial of $\tilde{\mathbf{R}}$, then $\Gamma(f) = \Gamma(g)$ implies that $\text{int}(f_x, f_y) = \text{int}(g_x, g_y)$. Thus the set of $f = y^n + a_2(x).y^{n-2} + \ldots + a_n(x) \in \tilde{\mathbf{R}}$ such that $\text{int}(f_x, f_y) = m$ is the union of equisingularity classes. We shall first prove that this union is finite. This is an immediate application of the next Proposition. Recall that if a subsemigroup of $\mathbf{Z}$ is minimally generated by $h+1$ elements, then h is called the *length* of the semigroup.

Proposition 4.1 *Let $h \in \mathbf{N}$ and consider a polynomial $f \in \tilde{\mathbf{R}}$ such that h is the length of $\Gamma(f)$. Let $\mu_{h+1} = \text{int}(f_x, f_y)$, and let r_h be the last generator of $\Gamma(f)$. We have the following:*

i) $h = 1$ implies that $r_h \geq 3$ and $\mu_{h+1} \geq 2$.

ii) $h = 2$ implies that $r_h \geq 13$ and $m_{h+1} \geq 16$.

iii) More generally:

1) $r_h \geq 12.4^{h-2} + \sum_{i=0}^{h-2} 4^i = \frac{5}{3}.2^{2h-1} - \frac{1}{3}$.

2) $\mu_{h+1} \geq 2 + 2.\sum_{i=0}^{h-2} 4^i + 12.\sum_{i=h-2}^{2h-4} 2^i = \frac{5}{3}.2^{2h} - 3.2^h + \frac{4}{3}$, assuming that the summation over negative exponents is 0.

Proof. i) In this case, by Lemma 1.8, $\mu_1 = (r_0 - 1).(r_1 - 1)$. Furthermore, $r_1 \geq 2$ and $r_0 \geq 2$; otherwise $\Gamma(f) =< 1 >$, and $h = 0$. On the other hand, $\gcd(r_0, r_1) = 1$. This shows that $\max(r_0, r_1) = r_1 \geq 3$ and $\mu_2 \geq 2$. Then our assertion follows. Note that $r_1 = 3$ and $\mu_2 = 2$ holds for $f = y^2 + ax^3$, where $a \in \mathbf{K} - 0$.

ii) Let g_2 be the second approximate root of f. Then $\mu_3 = d_2.\text{int}(g_{2_x}, g_{2_y}) + (d_2 - 1)(r_2 - 1)$. It follows from i) that $\frac{r_1}{d_2} \geq 3$ and that $\text{int}(g_{2_x}, g_{2_y}) = (\frac{r_0}{d_2} - 1)(\frac{r_1}{d_2} - 1) \geq 2$, and also that $\frac{r_0}{d_2} + \frac{r_1}{d_2} \geq 5$. In particular,

$$(r_0 - d_2)(\frac{r_1}{d_2} - 1) \geq 2.d_2$$

and

$$r_0 + r_1 \geq 5.d_2.$$

Thus,

$$r_1.\frac{r_0}{d_2} \geq d_2 + r_1 + r_0 \geq 6.d_2 \geq 12.$$

But $r_2 - 1 \geq r_1.\frac{d_1}{d_2} = r_1.\frac{r_0}{d_2}$. Finally, $r_2 \geq r_1\frac{r_0}{d_2} + 1 \geq 13$, and $\mu_{h+1} \geq 2.d_2 + (r_2 - 1) \geq 4 + 12 = 16$. This implies our assertion. Note that the lower bounds 13 and 16 are sharp: they are satisfied for $f = (y^2 + a.x^3)^2 + b.x^5y$, where $a, b \in \mathbf{K} - 0$, whose semigroup is $\Gamma =< 4, 6, 13 >$.

iii) We prove the inequalities by induction on h. From i) and ii) both are satisfied for $1 \leq h \leq 2$. Assume that $h \geq 3$ and that the formulas are true for $h - 1$. We first prove inequality 1). First, note that $r_h \geq (\frac{r_{h-1}}{d_h}).d_{h-1} + 1$. The quotient $\frac{r_{h-1}}{d_h}$ being the last generator of $\Gamma(g_h)$ which is of length $h - 1$, it follows by induction that $\frac{r_{h-1}}{d_h} \geq 12.4^{h-3} + \sum_{i=0}^{h-3} 4^i = \frac{5}{3}.2^{2h-3} - \frac{1}{3}$. On the other hand, $d_{h-1} \geq 4$, thus $r_h \geq 4.(\frac{5}{3}.4^{2h-3} - \frac{1}{3}) + 1 = \frac{5}{3}.2^{2h-1} - \frac{1}{3}$.
We now prove inequality 2). Consider the last approximate root g_h of f. We have $\mu_{h+1} = d_h.\text{int}(g_{h_x}, g_{h_y}) + (d_h - 1)(r_h - 1)$. But $d_h \geq 2$ and $r_h \geq \frac{5}{3}.2^{2h-1} - \frac{1}{3}$. On the other hand, the length of $\Gamma(g_h)$ being $h - 1$, it follows that

$$\text{int}(g_{h_x}, g_{h_y}) \geq 2 + 2.\sum_{i=0}^{h-3} 4^i + 12.\sum_{i=h-3}^{2h-6} 2^i = \frac{5}{3}.2^{2h-2} - 3.2^{h-1} + \frac{4}{3}$$

In particular,

$$\mu_{h+1} \geq \frac{5}{3}.2^{2h-2} - 3.2^{h-1} + \frac{4}{3} + \frac{5}{3}.2^{2h-1} - \frac{1}{3} - 1 = \frac{5}{3}.2^{2h} - 3.2^h + \frac{4}{3}$$

Remark 4.2 The bounds in the above Proposition are sharp. More precisely, for all $h \geq 1$, there is a polynomial $f_h(x, y) \in \tilde{R}$ such that h is the length of $\Gamma(f)$, and that $\text{int}(f_{hx}, f_{hy}) = \frac{5}{3}.2^{2h} - 3.2^h + \frac{4}{3}$, and if r_h denotes the last generator of $\Gamma(f)$, then $r_h = \frac{5}{3}.2^{2h-1} - \frac{1}{3}$. Consider the semigroup Γ_h generated by $r_0 = 2^h$ and

$$r_k = 2^{h-k}(\frac{5}{3}.2^{2k-1} - \frac{1}{3})$$

for all $1 \leq k \leq h$ (equivalently, $r_1 = 2^{h-1}.3, r_2 = 2^h.3 + 2^{h-2}, \ldots, r_{k+2} = 2^{h+k}.3 + \sum_{i=1}^{k+1} 2^{h+k-2i}$ for all $1 \leq k \leq h - 2$). Clearly $r_h = \frac{5}{3}.2^{2h-1} - \frac{1}{3}$, and the d-sequence is given by $d_k = 2^{h+1-k}, 1 \leq k \leq h + 1$. Furthermore,

$r_k d_k < r_{k+1} d_{k+1}$ for all $1 \leq k \leq h$. It follows that Γ_h is the semigroup of a polynomial of $\tilde{R}$. We shall prove by induction that the Milnor number of such a polynomial is $\frac{5}{3}.2^{2h} - 3.2^h + \frac{4}{3}$. Denote this number by μ_{h+1} and recall that $\mu_{h+1} = \sum_{k=1}^h (\frac{d_k}{d_{k+1}} - 1) r_k - r_0 + 1$. Since $\frac{d_k}{d_{k+1}} = 2$ for all $1 \leq k \leq h$, we have:

$$\mu_{h+1} = (\sum_{k=1}^h r_k) - r_0 + 1 = \sum_{k=1}^h 2^{h-k}(\frac{5}{3}.2^{2k-1} - \frac{1}{3}) - 2^h + 1$$

This is nothing but $\frac{5}{3}.2^{2h} - 3.2^h + \frac{4}{3}$ which proves our assertion.

Corollary 4.3 *Let $m \in 2.\mathbf{N}$, then one can effectively compute the set of irreducible polynomials $f \in \tilde{\mathbf{R}}$ such that $m = \mathrm{int}(f_x, f_y)$.*

Proof. It follows from Proposition 4.1 that the length h of the semigroup of a polynomial $f \in \tilde{\mathbf{R}}$ with $m = \mathrm{int}(f_x, f_y)$ takes a finite number of values. In fact, easy calculations show that h satisfies the inequality $2^h \leq M = \frac{9 + \sqrt{1 + 60m}}{10}$. In particular, $h \leq \frac{\ln(M)}{\ln(2)}$. Let $H = \{h \in \mathbf{N}; h \leq \frac{\ln(M)}{\ln(2)}\}$. Given $h \in A$ we shall effectively construct the set Σ of all the sequences $(r_0, r_1, \ldots, r_h)$ which minimally generate a semigroup of a polynomial $f \in \tilde{\mathbf{R}}$ of the required Milnor number. The steps of the algorithm can be summarized as follows:

Set $m = \mu_{h+1}$. We want to calculate the set of (μ_h, r_h, d_h) with the following equality:

(E1) $\mu_{h+1} = \mu_h d_h + (r_h - 1)(d_h - 1)$

Recall the following restrictions:

i) $d_h \geq 2$

ii) $r_h \geq \frac{5}{3}.2^{2h-1} - \frac{1}{3}$.

iii) $\gcd(r_h, d_h) = 1$.

iv) $\mu_h \geq \frac{5}{3}.2^{2h-2} - 3.2^{h-1} + \frac{4}{3}$

v) $\mu_1 = 0$, and for all $h \geq 2$, $\mu_h = \frac{\mu_{h+1} - (d_h - 1)(r_h - 1)}{d_h}$ is an even integer.

Now equality (E1) gives $(d_h - 1)(r_h - 1) = \mu_{h+1} - \mu_h d_h$, and by iv)

$$-\mu_h d_h \leq -[\frac{5}{3}.2^{2h-2} - 3.2^{h-1} + \frac{4}{3}]d_h.$$

In particular,

$$(r_h - 1)(d_h - 1) \leq \mu_{h+1} - [\frac{5}{3}.2^{2h-2} - 3.2^{h-1} + \frac{4}{3}].d_h$$

This gives us the following upper bound for r_h:

$$r_h \leq \frac{\mu_{h+1}}{d_h - 1} - [\frac{5}{3}.2^{2h-2} - 3.2^{h-1} + \frac{4}{3}].\frac{d_h}{d_h - 1} + 1$$

Corollary 4.4 *The above equality with (ii) give:*

$$(E2) \quad \frac{5}{3}.2^{2h-1} - \frac{1}{3} \leq r_h \leq \frac{\mu_{h+1}}{d_h - 1} - [\frac{5}{3}.2^{2h-2} - 3.2^{h-1} + \frac{4}{3}].\frac{d_h}{d_h - 1} + 1$$

In the following we shall refine the lower bound of Corollary 4.4. We start with the following lemma:

Lemma 4.5 $\sum_{i=1}^{h}(e_i - 1)r_i = r_h d_h - m_h$. *In particular,* $\mu_{h+1} = \sum_{i=1}^{h}(e_i - 1)r_i - r_0 + 1 = r_h d_h - m_h - r_0 + 1$, *where we recall that* $e_i = \dfrac{d_i}{d_{i+1}}$ *for all* $1 \leq i \leq h$.

Proof. Applying identity (**) of Section 3 with $k = h$ we get:

$$\sum_{i=1}^{h-1}(e_i - 1)r_i = r_h - m_h$$

Now adding $(e_h - 1)r_h = (d_h - 1)r_h$ to the equality establishes our assertion.
Lemma 4.5 and equality (E1) imply that $r_h d_h = \mu_{h+1} + m_h + r_0 - 1$. On the other hand, with the notation $m_0 = r_0$, we have for all $1 \leq k \leq h$, $m_k - m_{k-1} \geq d_{k+1}$. Adding these inequalities we get:

$$\mu_h \geq m_0 + d_2 + \ldots + d_h + d_{h+1} = d_1 + d_2 + \ldots + d_h + 1.$$

But for all $1 \leq k \leq h, d_k \geq 2^{h-k}.d_h$ and so $\mu_h \geq d_h.(2^h - 1) + 1$. Since $r_0 = d_1 \geq 2^{h-1}.d_h$, we have

$$r_h \geq \quad \max(\frac{5}{3}.2^{2h-1} - \frac{1}{3}, \frac{\mu_{h+1}}{d_h} + (3.2^{h-1} - 1))$$

Now equality (E1) implies that $\dfrac{\mu_{h+1}}{d_h} = \mu_h + (r_h - 1)(1 - \dfrac{1}{d_h})$. But $d_h \geq 2$ and so, using the inequalities of Proposition 4.1a, we get:

$$\frac{\mu_{h+1}}{d_h} \geq (\frac{5}{3}.2^{2h-2} - 3.2^{h-1} + \frac{4}{3}) + (\frac{5}{3}.2^{2h-1} - \frac{4}{3}).\frac{1}{2} = \frac{5}{3}.2^{2h-1} - 3.2^{h-1} + \frac{2}{3}$$

In particular, $\max(\dfrac{5}{3}.2^{2h-1} - \dfrac{1}{3}, \dfrac{\mu_{h+1}}{d_h} + (3.2^{h-1} - 1)) = \dfrac{\mu_{h+1}}{d_h} + (3.2^{h-1} - 1)$.
It follows that:

$$(E3) \quad \frac{\mu_{h+1}}{d_h} + (3.2^{h-1} - 1) \leq r_h \leq \frac{\mu_{h+1}}{d_h - 1} - [\frac{5}{3}.2^{2h-2} - 3.2^{h-1} + \frac{4}{3}].\frac{d_h}{d_h - 1} + 1.$$

We now use inequality (E3) to give an upper bound for d_h (a lower bound being 2). Note that

$$\frac{\mu_{h+1}}{d_h - 1} - [\frac{5}{3}.2^{2h-2} - 3.2^{h-1} + \frac{4}{3}].\frac{d_h}{d_h - 1} + 1 - (\frac{\mu_{h+1}}{d_h} + (3.2^{h-1} - 1)) \geq 0.$$

If we set $p = (\frac{5}{3}.2^{2h-2} - 3.2^{h-1} + \frac{4}{3})$ and $q = 3.2^{h-1} - 2$, then an obvious analysis of the above inequality shows that it is equivalent to saying that $(p+q).d_h^2 - qd_h - \mu_{h+1} \leq 0$, which is true if and only if the following holds:

$$\text{(E4) } 2 \leq d_h \leq \frac{q + \sqrt{q^2 + 4\mu_{h+1}.(p+q)}}{2.(p+q)}$$

$$= \frac{3.2^{h-1} - 2 + \sqrt{(3.2^{h-1} - 2)^2 + 4\mu_{h+1}.(\frac{5}{3}.2^{2h-2} - \frac{2}{3})}}{\frac{10}{3}.2^{2h-2} - \frac{4}{3}}$$

The algorithm: The two integers μ_{h+1} and h being fixed, inequality (E4) determines the set D^h of possible values of d_h. Each value of d_h gives rise, using inequality (E3), to a set, denoted by $R^h_{d_h}$, of possible values of r_h (Note that $\frac{\mu_{h+1} - (d_h - 1)(r_h - 1)}{d_h}$ should be an even integer). We obtain the set, denoted by $P^h_{d_h}$, of possible values of (μ_h, r_h, d_h). Now we restart with the set of μ_h. This process shall stop after it has constructed a set of lists of length h. The set of semigroups corresponding to μ_{h+1} is a subset of this list and can be easily calculated. Note that if $h = 1$, then $\mu_1 = 0$ and $\mu_2 = (r_1 - 1)(d_1 - 1)$ by condition v). In this case, the values of $(r_1, d_1 = r_0)$ can also be obtained from the set of divisors of μ_2.

Example 4.6 We perform an explicit computation for $\mu_{h+1} = 28$. In this case, $M = \frac{9 + \sqrt{1 + 60.28}}{10} = 5$, so $H = \{h : 1 \leq h \leq \frac{\ln(5)}{\ln(2)}\} = \{1, 2\}$.

1) $h = 1$: In this case, since $28 = 1 * 28 = 2 * 14 = 4 * 7$, we have $(r_1, d_1) \in \{(2, 29), (3, 15), (5, 8)\}$ and condition iii) eliminates $(3, 15)$. We get the semigroups $< 2, 29 >$ and $< 5, 8 >$. The canonical representative of the equisingularity class of the first one (resp. the second one) is $y^2 - x^{29}$ (resp. $y^5 - x^8$).

2) $h = 2$: Inequality (E4) implies in this case that $2 \leq d_2 \leq \frac{4 + \sqrt{688}}{12} < 3$. In particular, $D^2 = \{2\}$.

Now inequality (E3) implies that $\frac{28}{2} + 5 = 19 \leq r_2 \leq 28 - 4 + 1 = 25$, and with conditions iii), v), we get $R^2_2 = \{21, 25\}$. If $r_2 = 25$ (resp. $r_2 = 21$), then $\mu_2 = 2$ (resp. $\mu_2 = 4$). Thus $P^2_2 = \{(2, 25, 2), (4, 21, 2)\}$.

i) $(\mu_2, r_2, d_2) = (2, 25, 2)$. Applying the construction above to $\mu_2 = 2$, we get $\frac{d_1}{d_2} = 2, \frac{r_1}{d_2} = 3$. This leads to the semigroup $< 4, 6, 25 >$. The canonical representative of the equisingularity class of this semigroup is $(y^2 - x^3)^2 - x^{11}y$.

ii) $(\mu_2, r_2, d_2) = (4, 21, 2)$. Applying the construction above to $\mu_2 = 4$, we get $\frac{d_1}{d_2} = 2, \frac{r_1}{d_2} = 5$. This leads to the semigroup $< 4, 10, 21 >$. The canonical representative of the equisingularity class of this semigroup is $(y^2 - x^5)^2 - x^8 y$.

Let m be an even integer, and let H is the set of positive integers not exceeding $\frac{\ln(M)}{\ln(2)}$, where $M = \frac{9 + \sqrt{1 + 60m}}{10}$. Assume that H is not reduced to 0 and let h be a nonzero element of H. Set $m = \mu_{h+1}$ and let

$$a_h = \frac{q + \sqrt{q^2 + 4\mu_{h+1}.(p + q)}}{2.(p + q)} = \frac{3.2^{h-1} - 2 + \sqrt{(3.2^{h-1} - 2)^2 + 4\mu_{h+1}.(\frac{5}{3}.2^{2h-2} - \frac{2}{3})}}{\frac{10}{3}.2^{2h-2} - \frac{4}{3}}.$$

Let D^h be the set positive integers between 2 and a_h (one can easily verify that the condition $a_h \geq 2$ is equivalent to the numerical condition $\mu_{h+1} \geq \frac{5}{3}.2^{2h} - 3.2^h + \frac{1}{3}$ proved in Proposition 4.1, in particular, D^h is not the empty set). Set $P^h = \bigcup_{d \in D^h} P_d^h$ and denote by C_{h+1} the cardinality of P^h. We shall give an upper bound for C_{h+1}. Set $b_d^h = \frac{\mu_{h+1}}{d} + (3.2^{h-1} - 1)$, and

$$c_d^h = \frac{\mu_{h+1}}{d-1} - [\frac{5}{3}.2^{2h-2} - 3.2^{h-1} + \frac{4}{3}].\frac{d}{d-1} + 1.$$

The set R_d^h of possible values of r_h is a subset of the set of integers between b_d^h and c_d^h. Its cardinality is then bounded by $c_d^h - b_d^h + 1$. Furthermore, we easily verify that if $r \in R_d^h$, then

$$\frac{\mu_{h+1} - (r-1)(d-1)}{d} \geq \frac{5}{3}.2^{2h-2} - 3.2^{h-1} + \frac{1}{3}.$$

In particular, if $h \geq 2$, then $(\mu_h = \frac{\mu_{h+1} - (r-1)(d-1)}{d}, r, d)$ is an element of P_d^h.
Now

$$c_d^h - b_d^h + 1 = \frac{\mu_{h+1}}{d(d-1)} - (p_h - q_h).\frac{d}{d-1} - q_h + 1$$

$$= \frac{\mu_{h+1}}{d-1} - \frac{\mu_{h+1}}{d} - (p_h - q_h).(1 + \frac{1}{d-1}) - q_h + 1$$

Consequently, if $a = [a_h]$, then the cardinality C_{h+1} of P_h is bounded by

$$\sum_{d=2}^{a}(c_d^h - b_d^h + 1) = (\mu_{h+1})(1 - \frac{1}{a}) - p_h(a-1) - (p_h - q_h).\sum_{d=1}^{a-1}\frac{1}{d} + (a-1)$$

But $1 - \frac{1}{a} < 1$, and substituting $a = 2$ we get:

$$C_{h+1} \leq \mu_{h+1} - 2p_h + q_h + 1 = \mu_{h+1} - (\frac{10}{3}.2^{2h-2} - 3.2^{h-1} - \frac{1}{3}) = \mu_{h+1} - (\frac{B_{h+1}}{2} - 1)$$

where $B_{h+1} = \frac{5}{3}.2^{2h} - 3.2^h + \frac{4}{3}$ is the lower bound of μ_{h+1} in Proposition 4.1.

Remark 4.7 Note that the bound above is not the optimal one. Indeed, given $d \in D^h$, the cardinality of the set of $r \in R_d^h$ such that $\gcd(r, d) = 1$ is bounded by $\frac{c_d^h - b_d^h}{d} + 1$, but in view of our algorithm, all values of R_d^h are used, in particular the value above bounds also the number of operations used in the first step of the algorithm.

Let (μ_h, r, d) be an element of P_h and recall that $\mu_h = \frac{\mu_{h+1}}{d} - (r-1)\frac{d-1}{d}$. Since $b_d^h \le r \le c_d^h$,

$$\mu_h \le \frac{\mu_{h+1}}{d} - (b_d^h - 1)\frac{d-1}{d} \le \frac{\mu_{h+1}}{d} - (b_d^h - 1)\frac{d-1}{d} = \frac{b_d^h - 1}{d} - (3.2^{h-1} - 2)$$

$$= \frac{\mu_{h+1}}{d^2} + (\frac{1}{d} - 1)(3.2^{h-1} - 2) \le \frac{\mu_{h+1}}{4} - \frac{1}{2}(3.2^{h-1} - 2) \le \frac{\mu_{h+1}}{4} - (3.2^{h-2} - 1)$$

Let $A_{h+1} = 3.2^{h-2} - 1$. It follows by induction that for all $0 \le k \le h-1$, if μ_{h-k} is a possible value of the Milnor number at the step $k+1$, then we have:

$$\mu_{h-k} \le \frac{\mu_{h+1}}{4^{k+1}} - \sum_{i=0}^{k} \frac{A_{h+1-i}}{4^{k-i}} = 3.2^{h-2k-2}(2^{k+1} - 1) - \frac{4}{3} - \frac{1}{3.4^k}$$

(Note that the above inequality is valid if $k = h-1$ because $\mu_1 = 0$). Thus, we obtain a bound of the set of values calculated at the step $k+1, 0 \le k \le h-2$ as follows: Let μ_{h-k} be a possible value of the Milnor number obtained by iterating the algorithm above $k+1$ times, and denote by $C_{h-k}(\mu_{h-k})$ the cardinality of the set and by $P_{h-k-1}(\mu_{h-k})$ the 3-uplets (μ, r, d) obtained by applying the algorithm above to μ_{h-k} instead of μ_{h+1}. It follows that

$$C_{h-k} \le \mu_{h-k} - \frac{B_{h-k}}{2} + 1 \le \frac{\mu_{h+1}}{4^{k+1}} - \sum_{i=0}^{k} \frac{A_{h+1-i}}{4^{k-i}} - \frac{B_{h-k}}{2} + 1$$

$$= \frac{\mu_{h+1}}{4^{k+1}} - 3.2^{h-k-2} + 3.2^{h-2k-2} - \frac{5}{3}.2^{2h-2k-1} - \frac{1}{3.4^k} + \frac{5}{3}$$

In particular, the cardinality of the set of semigroups corresponding to the given Milnor number $m = \mu_{h+1}$ is bounded by $\prod_{i=2}^{h} C_{h+1-i}$ which is a polynomial in m bounded by its leading coefficient $\frac{m^h}{2^{h(h-1)}}$. Note that, in view of Remark 4.8, the number of operations used in the algorithm is then bounded by $\sum_{i=0}^{h-1} \prod_{k=0}^{i} C_{h+1-k}$.

The above algorithm has been implemented with *MAPLE*. The input is an integer m, and the output is the list of semigroups whose conductor is m. In the implementation work we followed the ideas explained above, with the following simplification: at the last step, the set of values we are interested in is calculated by using the factorization of the given Milnor number. The algorithm is an iteration of the following:

Input: $m \in 2.\mathbf{N}$

Output: The set P^h.

Step I: Compute the set H.

Step II: Take $h \in H$.

Step III: Compute the set D^h.

Step IV: Take $d \in D^h$.

Step V: Compute R_d^h

(*) if $r \in R_d^h$ *and* $gcd(r, d) = 1$ *and* $\dfrac{m-(d-1)(r-1)}{d} \in 2.\mathbf{N}$ *(resp.* $(d-1)(r-1) = m$ *if h=1) then add* $(\dfrac{m-(d-1)(r-1)}{d}, r, d)$ *to* P_d^h.

Step VI: $P^h = \bigcup_{d\in D^h} P_d^h$

The main operation of the algorithm is the one described in line (*). We experimented with it on various values of m: the computation took about 0.2 sec for $m = 160$, 0.7 sec for $m = 300$, 1.5 sec for $m = 500$, and 3 sec for $m = 1000$.

```
> ordc:=proc(L::list)
> ## Rewrite a sequence as an increasing one
> local  oL, i, j,n, x; n:=nops(L); oL:=L;
> if n=1 then L; fi; for i from 1 to n-1 do
> for j from i+1 to n do
> if oL[i]> oL[j] then
> x:=oL[i];
> oL[i]:=oL[j];
> oL[j]:=x;
> fi
> od
> od; [op(oL)] end:

> ordd:=proc(L::list)
> ##Rewrite a sequence as a decreasing one
> local  oL, i, j,n, x; n:=nops(L); oL:=L;
> if n=1 then L; fi; for i from 1 to n-1 do
> for j from i+1 to n do
> if oL[i]<oL[j] then
> x:=oL[i];
> oL[i]:=oL[j];
> oL[j]:=x;
> fi
> od
> od; [op(oL)] end:

> ordc([3,1,2]); ordd([1,2,3]);ordd([1]);
```

$$[1, 2, 3]$$
$$[3, 2, 1]$$
$$[1]$$

```
> Xprod:=proc()
> ## Gives the cartesian product of given sets
> local S,s,n,k,x,l; S:={[]};n:=nargs; for
> k to n do s:=NULL; for x in args[k] do for l in S do
> l:=[op(l),x];s:=s,l; od;od; S:={s};
> od;
> S
> end:
```

```
> h:=proc(L::list)
> ## Gives the gcd sequence and the e sequence
> local i,k,l,n,x,F,G,d,D,R,E,c; n:=nops(L);
> l:=L; if n=1 and L[1]<> 1 then RETURN(false) fi; if n=1 and L[1]=1
> then RETURN([1]) fi;
> d[1]:=L[1]; for k from 2 to n do d[k]:=igcd(d[k-1],L[k]); od;
> d:=[seq(d[k],k=1..nops(L))];
> for k from 1 to nops(L)-1 do
> if d[k]=d[k+1] then
> x:=d[k];
> d[k+1]:=x;
> x:='x';
> x:=l[k];
> l[k+1]:=x;
> else fi od;
> F:={op(d)}; G:={op(l)}; D:=ordd([op(F)]);
> if op(nops(D),D) > 1 then RETURN(false);fi; R:=[op(G)]; if nops(D)=1
> and D[1]=1 then  RETURN([1]);fi; c:=(nops(D)-1); for k to c do
> E[k]:=D[k]/(D[k+1]); od; E:=[seq(E[k],k=1..c)]; if c=1 then
> RETURN([D,R,E]);fi; for k to nops(E)-1 do if E[k]*R[k+1] >= R[k+2]
> then false;
> else  [D,R,E] fi; od;  end:
```

```
> h([4,6,13]);
> h([2,3]);h([4,6,17]);h([4]);h([1]);h([1,2,4]);h([4,6,8]);h([2,4,7]);h(
> [4,6]);
```

$$[[4, 2, 1], [4, 6, 13], [2, 2]]$$

$$[[2, 1], [2, 3], [2]]$$

$$[[4, 2, 1], [4, 6, 17], [2, 2]]$$

$$\mathit{false}$$

$$[1]$$

$$[1]$$

$$\mathit{false}$$

$$[[2, 1], [2, 7], [2]]$$

$$\mathit{false}$$

```
> s:=proc(L::list) local
> k,n,A,B,E,T,P,V,i,P1,E1,N,O,R,S,W,j,M,Q1,G,a,p;
> if h(L)= false then false
> elif h(L)=[1] then Y
> else
> A:=op(1,h(L));
> ## A=(d1,d2,...,dh,1): The gcd-sequence
> B:=op(2,h(L));
> ## B=(r0,r1,...,rh): The r-sequence
> E:=op(3,h(L));
> ## E=(e1,...eh): The e-sequence
> n:=nops(E);
> ##n=h: The length of the semigroup
> for k to n do
> E1[k]:={op([seq(i,i=0..E[k]-1)])};
> od;
> E1:=[0,seq(E1[k],k=1..n)];
> ## E1=[0,{0,1,...,e1-1},....{0,1,...,eh-1}]
> T:=[seq(B[k+1]*E(k)/B[1],k=1..nops(E))];
> ## T:=[rkek/r0,k=1...h]
> P:=[seq(trunc(op(1,op(k,T))),k=1..nops(T))];
> ##P=[[rkek/r0],k=1,...,h]: bounds of theta(0), coefficients of r0
> for k to n do
> P1[k]:={op([seq(i,i=0..P[k])])};
> ## The set {0,1,...,P[k]}
> od;
> P1:=[seq(P1[k],k=1..n)];
> ##The sequence of sets P1[k], k=1,...,h
> V[1]:={op(P1[1])};
> ## V[1]={0,1,...,[r1e1/r0]}: bounds of theta(0) in: r1e1/d2=theta(0).r0/d2
> for k from 2 to nops(P1) do
> V[k]:=[{op(P1[k])},op([seq({op(E1[i])},i=2..k)])];
> od;
> V:=[seq(V[k],k=1..nops(P1))];
> ## V[k]=[P1[k],E1[2],...,E1[k]]
> N[1]:=P1[1];
> ##N[1]=The set of possible values of theta_0^1
> for k from 2 to nops(P1) do
> N[k]:=Xprod(op(V[k]));
> ## N[k]=The set of coefficients of O[k]
> od;
> N:=[seq(N[k],k=1..nops(P1))];
> for k to nops(E) do
> O[k]:=B[k+1]*E[k]/A[k+1];
> od;
> O:=[seq(O[k],k=1..nops(E))];
> for k to nops(E) do
> R[k]:=[seq(B[i]/A[k+1],i=1..k)];
> od;
> R:=[seq(R[k],k=1..nops(E))];
> for j to nops(N[1]) do
> if op(O[1])=op(R[1])*op(j,N[1])
> then S[1]:=[op(j,N[1])]; else fi;
> od;
> for k from 2 to nops(R) do for j to nops(N[k]) do
> if op(O[k])=sum('op(i,R[k])*op(i,op(j,N[k]))','i'=1..nops(R[k]))
> then S[k]:=[seq(op(i,op(j,N[k])),i=1..nops(R[k]))]; else fi;
> od;
> od;
> S:=[seq(S[k],k=1..nops(R))];
> G[0]:=X; G[1]:=Y; G[2]:=G[1]^E[1]-G[0]^op(S[1]);
> for k from 2 to n do
> G[k+1]:=G[k]^E[k]-product(G[p]^op(p+1,S[k]),p=0..k-1);
> od;
> G:=[seq(G[k],k=1..n+1)];
> ##The set of Approximate Roots
> B,A,E ,O,R, n,S,G fi;
> ## The r-sequence, The d-sequence, The e-sequence, The sequence of
coefficients, The G-sequence
> end:
```

```
> s([8,12,26,53]);
```

$[8, 12, 26, 53], [8, 4, 2, 1], [2, 2, 2], [6, 26, 106], [[2], [4, 6], [8, 12, 26]], 3,$
$[[3], [5, 1], [10, 0, 1]],$
$[Y, Y^2 - X^3, (Y^2 - X^3)^2 - X^5\, Y, ((Y^2 - X^3)^2 - X^5\, Y)^2 - X^{10}\, (Y^2 - X^3)]$

```
> s([4,6,7]);s([4,5]);
```

false

$$[4, 5], [4, 1], [4], [20], [[4]], 1, [[5]], [Y, Y^4 - X^5]$$

```
> s([4,6,13]);
```

$[4, 6, 13], [4, 2, 1], [2, 2], [6, 26], [[2], [4, 6]], 2, [[3], [5, 1]],$
$[Y, Y^2 - X^3, (Y^2 - X^3)^2 - X^5\, Y]$

```
> s([1]);s([1,2,4]);s([4,6,8]);
```

Y

Y

false

```
> h([1,2,4]);s([1,2,4]);s([4,6]);
```

$[1]$

Y

false

```
> h([3,5]);s([3,5]);
```

$$[[3, 1], [3, 5], [3]]$$

$$[3, 5], [3, 1], [3], [15], [[3]], 1, [[5]], [Y, Y^3 - X^5]$$

```
> T:=proc(L)
## Tests if a list is the semigroup of an irreducible polynomial
> local k,d; d[1]:=L[1];
> for k to nops(L)-1 do
> d[k+1]:=igcd(d[k],L[k+1]); od;
> d:=[seq(d[k],k=1..nops(L))];
> if nops(L)<=2 then L; else
> for k to nops(L)-2  do
> if d[k]*L[k+1]>=d[k+1]*L[k+2] then RETURN(false);
> else  L       fi; od; fi; end:
```

```
> H:=proc(m)
> ## Computes the set of possible h
> local i,c, H; c:=trunc(evalf(ln((9+sqrt(1+60*m))/10)/ln(2)));
> H:=[seq(i,i=1..c)]; end:
```

```
> C:=proc(L::list)
> ## gives the first step of the calculation when h=1
> local
> m,h,k,p,q,D,P,i,d,r,R,a,b,m1,M1,M,N,s,F;
> m:=op(1,L);
> if m=0 then L;
> else
> D:=trunc(evalf((1+sqrt(1+4*m))/(2)));
> if D=1 then print('smooth') else
> P:={op([seq(i,i=2..D)])}; fi;
> N:={}; for d in P do
> a:=trunc(evalf(m/d+1));
> b:=trunc(evalf((m/(d-1))+1));
> R:={op([seq(i,i=a..b)])}; M:={}; for r in R   do if
> (m=(r-1)*(d-1) and igcd(d,r)=1) then
> M:=M union{[0,d*op(2,L),r*op(2,L)]}; else fi; od; N:=N union
M;
> od;  fi;N end:
```

```
> M:=proc(L::list,h)
> ## gives the first step in the calculation when h > 1
> local
> m,k,p,q,D,P,i,d,r,R,a,b,c,m1,M1,M,N,s,H,F;
> m:=op(1,L); H:=H(m);
> if m=0 then L;
> else
> N:={};
> if h=1 then N:=C(L);  else p:=(5/3)*2^(2*h-2)-3*2^(h-1)+(4/3);
> q:=3*2^(h-1)-2;
> D:=trunc(evalf((q+sqrt(q^2+4*m*(p+q)))/(2*(p+q))));
> if D=1 then print('smooth') else
> P:={op([seq(i,i=2..D)])}; fi;
> for d in P do
> a:=trunc(evalf(m/d+q));
> b:=trunc(evalf((m/(d-1))-p*(d/(d-1))+1));
> R:={op([seq(i,i=a..b)])}; M:={}; for r in R   do
> m1:=(m-(r-1)*(d-1))/d; if (m1=trunc(m1) and m1 mod 2 = 0 and
> igcd(d,r)=1) then
> M:=M union{[m1,d*op(2,L),r*op(2,L)]}; else fi; od; N:=N union
M;
> od; fi;fi; N end:
```

```
> B:=proc(m,h)
> ## Gives the list containing the semigroups of length h
> local
> k,L,H,j,i,K,D,S,X,r,Y,N,Na,Xa,Ka,Ra,R,Ya,Z,n;
> if m mod 2=1 then RETURN([ ]) fi; X:=[[m,1,m]]; n:=h;
> Z:=[op(M(X[1],n))]; K:={op([seq(op(1,op(j,Z)),j=1..nops(Z))])};
> while K<>{0}  and  n> 1 do n:=n-1;
> for j to nops(Z) do
> Xa[j]:=[op(M(Z[j],n))];
> for i to nops(Xa[j]) do
> Y[j][i]:=[op(Xa[j][i]),op([seq(op(k,op(j,Z)),k=3..nops(Z[j]))])];
> od;
> Y[j]:=op([seq(Y[j][i],i=1..nops(Xa[j]))]); od;
> Y:=[seq(Y[j],j=1..nops(Z))];
> Ka:={op([seq(op(1,op(j,Y)),j=1..nops(Y))])};
> X:=Y;
> Z:=X;K:=Ka; Y:='Y';
> od;
> Z
> end:

> B(160,2);
```

[[0, 4, 78, 85], [0, 6, 40, 85], [0, 4, 74, 89], [0, 6, 38, 89], [0, 8, 26, 89], [0, 4, 70, 93],
[0, 4, 66, 97], [0, 6, 34, 97], [0, 10, 18, 97], [0, 4, 62, 101], [0, 6, 32, 101],
[0, 8, 22, 101], [0, 12, 14, 101], [0, 4, 58, 105], [0, 10, 16, 105], [0, 4, 54, 109],
[0, 6, 28, 109], [0, 4, 50, 113], [0, 6, 26, 113], [0, 8, 18, 113], [0, 10, 14, 113],
[0, 4, 46, 117], [0, 4, 42, 121], [0, 6, 22, 121], [0, 10, 12, 121], [0, 4, 38, 125],
[0, 6, 20, 125], [0, 8, 14, 125], [0, 4, 34, 129], [0, 4, 30, 133], [0, 6, 16, 133],
[0, 4, 26, 137], [0, 6, 14, 137], [0, 8, 10, 137], [0, 4, 22, 141], [0, 4, 18, 145],
[0, 6, 10, 145], [0, 4, 14, 149], [0, 6, 8, 149], [0, 4, 10, 153], [0, 4, 6, 157],
[0, 8, 20, 49], [0, 10, 25, 36]]

```
> S:=proc(m,h)
> ## gives the set of semigroups of length h
> local k, L,j,S,V;
> L:=B(m,h);
> for j to nops(L) do
> V[j]:=subsop(1=NULL,L[j]); od;
> V:=[seq(V[j],j=1..nops(L))];
> S:={op([seq(T(V[j]),j=1..nops(L))])}minus{false};
> S end:

> F:=proc(m)
> ## gives the set of semigroups
> local
> k,L,SG; if m mod 2 = 1 then RETURN([ ]) fi;
> L:=H(m);
> for k to nops(L) do
> SG[k]:=[k,S(m,k)]; od;
> SG:=[seq(SG[k],k=1..nops(L))]; end;
```

$F := \mathbf{proc}(m)$
$\mathbf{local}\, k,\ L,\ SG;$
$\quad \mathbf{if}\, m \bmod 2 = 1\, \mathbf{then}\ \ \mathrm{RETURN}([])\, \mathbf{end\ if};$
$\quad L := \mathrm{H}(m)\,;$
$\quad \mathbf{for}\, k\, \mathbf{to}\, \mathrm{nops}(L)\, \mathbf{do}\, SG_k := [k,\ \mathrm{S}(m,\ k)]\, \mathbf{end\ do}\,;$
$\quad SG := [\mathrm{seq}(SG_k,\ k = 1..\mathrm{nops}(L))]$
$\mathbf{end\ proc}$

```
> H(84);F(84);
```

[1, 2, 3]

[[1, {[2, 85], [3, 43], [4, 29], [5, 22], [7, 15], [8, 13]}], [2, {[6, 16, 57], [4, 26, 61], [6, 14, 61], [8, 10, 61], [4, 22, 65], [4, 18, 69], [6, 10, 69], [4, 14, 73], [6, 8, 73], [4, 10, 77], [4, 6, 81], [6, 15, 37], [6, 9, 40]}], [3, {[8, 12, 26, 53]}]]

```
> F(16);
```

[[1, {[2, 17]}], [2, {[4, 6, 13]}]]

```
> F(2);
```

[[1, {[2, 3]}]]

```
> F(1);
```

[]

```
> F(7);F(9);
```

[]

[]

```
> F(160);
```

[[1, {[11, 17], [2, 161], [5, 41]}], [2, {[10, 18, 97], [6, 32, 101], [8, 22, 101], [12, 14, 101], [10, 16, 105], [6, 28, 109], [4, 18, 145], [6, 10, 145], [4, 14, 149], [6, 8, 149], [4, 10, 153], [4, 6, 157], [8, 20, 49], [6, 16, 133], [4, 26, 137], [6, 14, 137], [8, 10, 137], [4, 22, 141], [4, 54, 109], [8, 18, 113], [6, 26, 113], [4, 50, 113], [10, 14, 113], [4, 46, 117], [10, 12, 121], [4, 42, 121], [6, 22, 121], [8, 14, 125], [6, 20, 125], [4, 38, 125], [4, 34, 129], [4, 30, 133]}], [3, {[8, 12, 42, 113], [8, 12, 38, 117], [8, 12, 34, 121], [8, 12, 30, 125], [8, 12, 26, 129], [8, 20, 46, 101], [8, 20, 42, 105], [8, 12, 50, 105], [8, 12, 46, 109]}]]

```
> H(1000);
```

[1, 2, 3, 4]

```
> H(500);
```

[1, 2, 3, 4]

```
> F(100);
```

[[1, {[5, 26], [2, 101]}], [2, {[10, 12, 61], [4, 30, 73], [8, 10, 77], [6, 14, 77], [4, 26, 77], [6, 8, 89], [4, 10, 93], [4, 6, 97], [4, 22, 81], [6, 10, 85], [4, 18, 85], [6, 16, 73], [8, 14, 65], [6, 20, 65], [4, 34, 69], [4, 14, 89]}],
[3, {[8, 12, 30, 65], [8, 12, 26, 69]}]]

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