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# Boundary Value Problems for Impulsive Functional Differential Equations with Infinite Delay

**M. Benchohra[1], Johnny Henderson[2], S.K. Ntouyas[3], A. Ouahab[1]**

[1] Laboratoire de Mathématiques, Université de Sidi Bel Abbès
BP 89, 22000 Sidi Bel Abbès, Algérie
e-mail: benchohra@univ-sba.dz; agh_ouahab@yahoo.fr

[2] Department of Mathematics, Baylor University
Waco, Texas 76798-7328, USA
e-mail: Johnny_Henderson@baylor.edu

[3] Department of Mathematics, University of Ioannina
451 10 Ioannina, Greece
e-mail: sntouyas@cc.uoi.gr

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**Abstract**

In this paper we investigate the existence of solutions to some classes of boundary value problems for impulsive functional and neutral functional differential equations with infinite delay, using the nonlinear alternative of Leray-Schauder type.

## 1 Introduction

This paper is concerned with the existence of solutions to first order boundary value problems for impulsive functional and neutral functional differential equations with infinite delay. In particular, in Section 3, we will consider the class of first order functional differential equations with impulsive effects,

$$y'(t) = f(t, y_t),\ a.e.\ t \in J := [0, \infty),\ t \neq t_k,\ k = 1, \dots, \tag{1}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)),\quad t = t_k,\ k = 1, \dots, \tag{2}$$

$$Ay(t) - y_\infty = \phi(t),\quad t \in (-\infty, 0], \tag{3}$$

where $f : J \times B \to \mathbb{R}^n,\ I_k : \mathbb{R}^n \to \mathbb{R}^n,\ k = 1, 2, \dots$ are given functions satisfying some assumptions that will be specified later, $\lim\limits_{t\to\infty} y(t) = y_\infty,\ A > 1,\ \phi \in B$ and $B$ is called a *phase space* that will be defined later. Section 4 is devoted to impulsive neutral functional differential equations,

$$\frac{d}{dt}[y(t) - g(t, y_t)] = f(t, y_t),\ t \in J,\ t \neq t_k,\ k = 1, \dots, \tag{4}$$

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$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad t = t_k, \ k = 1, \dots, \tag{5}$$

$$Ay(t) - y_\infty = \phi(t), \ t \in (-\infty, 0], \tag{6}$$

where $f, I_k, y_\infty, B$ are as in problem (1)-(3), and $g : J \times B \to \mathrm{I\!R}^n$ is a given function. The notion of the phase space $B$ plays an important role in the study of both qualitative and quantitative theory. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [16] (see also Kappel and Schappacher [18] and Schumacher [26]). For a detailed discussion on this topic we refer the reader to the book by Hino *et al* [17]. For the case where the impulses are absent (i.e $I_k = 0, \ k = 1, \dots, m$), an extensive theory has been developed for the problem (1)–(3). We refer to Hale and Kato [16], Corduneanu and Lakshmikantham [7], Hino *et al* [17], Lakshmikantham *et al* [20] and Shin [27].

Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in control, physics, chemistry, population dynamics, biotechnology and economics. There has been a significant development in impulse theory, in recent years, especially in the area of impulsive differential equations with fixed moments; see the monographs of Bainov and Simeonov [3], Lakshmikantham *et al* [19] and Samoilenko and Perestyuk [25] and the papers of Agur *et al* [1], Ballinger and Liu [4], Benchohra *et al* [5, 6], Franco *et al* [9] and the references therein.

Boundary value problems on infinite intervals appear in many problems of practical interest, for example in linear elasticity problems, nonlinear fluid flow problems and foundation engineering (see [2, 10, 11, 21, 22, 23, 24]) and the references therein. Recently the fixed point argument such as the Banach contraction principle, fixed point index theory and monotone iterative technique were applied to first and second order impulsive differential equation. We mention here the papers by Guo [12, 13, 14, 15], Yan and Liu [28] and the references therein.

Our goal here is to give existence results for the above problems by using the nonlinear alternative of Leray-Schauder type. These results can be considered as a contribution to the literature.

## 2 Functional Differential Equations

Let $C([0, \infty), \mathrm{I\!R}^n)$ be the space of all continuous functions from $[0, \infty)$ into $\mathrm{I\!R}^n$.
$L^1([0, \infty), \mathrm{I\!R}^n)$ denotes the Banach space of measurable functions $y : [0, \infty) \longrightarrow \mathrm{I\!R}^n$ which are Lebesgue integrable and normed by

$$\|y\|_{L^1} = \int_0^\infty |y(t)| dt \quad \text{for all} \quad y \in L^1([0, \infty), \mathrm{I\!R}^n).$$

**Definition 2.1** *The map $f : [0, \infty) \times B \longrightarrow \mathrm{I\!R}^n$ is said to be $L^1$-Carathéodory if*

*(i) $t \longmapsto f(t, y)$ is measurable for each $y \in B$;*

*(ii)* $y \longmapsto f(t,y)$ *is continuous for almost all* $t \in [0,b]$;

*(iii) For each* $q > 0$, *there exists* $h_q \in L^1([0,\infty), \mathbb{R}_+)$ *such that*

$$\|f(t,y)\| \leq h_q(t) \quad \textit{for all} \quad \|y\|_B \leq q \quad \textit{and for almost all} \quad t \in [0,\infty).$$

In order to define the phase space and the solutions of (1)–(3) we shall consider the space

$$\begin{aligned} B_* \; &= \Big\{ y : (-\infty,\infty) \to \mathbb{R}^n, \;\; \text{such that,} \;\; y(t_k^-), y(t_k^+), \;\text{exist with}\;\; y(t_k) = y(t_k^-), \\ & y(t) = \phi(t), t \leq 0, \;\; y_k \in C(J_k, \mathbb{R}^n), \;\; k = 1, \ldots \Big\}, \end{aligned}$$

where $y_k$ is the restriction of $y$ to $J_k = (t_k, t_{k+1}], \; k = 0, \ldots$. Here $t_0 = 0$. Set

$$B_b = \{ y \in B_* : \sup_{t \in J} |y(t)| < \infty \}.$$

Let $\|\cdot\|_b$ be the seminorm in $B_b$ defined by

$$\|y\|_b = \|y_0\|_B + \sup\{|y(s)| : 0 \leq s < \infty\}, \;\; y \in B_b.$$

We will assume that $B$ satisfies the following axioms:

(A) If $y : (-\infty,\infty) \to \mathbb{R}^n$, and $y_0 \in B$, then for every $t \in [0,\infty)$ the following conditions hold:

(i) $y_t$ is in $B$;

(ii) $\|y_t\|_B \leq K(t) \sup\{|y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_B$,

where $H \geq 0$ is a constant, $K : [0,\infty) \to [0,\infty)$ is continuous, $M : [0,\infty) \to [0,\infty)$ is locally bounded and $H, K, M$ are independent of $y(\cdot)$.

(A-1) For the function $y(\cdot)$ in $(A)$, $y_t$ is a $B$-valued continuous function on $[0,\infty)$.

(A-2) The space $B$ is complete.

Let us define what we mean by a solution of problem (1)–(3).

**Definition 2.2** *A function* $y \in B_b$, *is said to be a solution of (1)–(3) if* $y$ *satisfies (1)–(3).*

In the proof of the existence results for the problem (1)-(3) we need the following auxiliary lemma.

**Lemma 2.1** *Let* $f : C([0,\infty), \mathbb{R}^n) \to \mathbb{R}^n$ *be continuous and* $\int_0^\infty f(s)ds < \infty$. *Then* $y$ *be a solution of the impulsive integral equation*

$$y(t) = \begin{cases} \dfrac{\phi(0)}{A(A-1)} + \dfrac{1}{A-1}\left[\displaystyle\int_0^\infty f(y_s)ds + \sum_{k=1}^{\infty} I_k(y(t_k))\right] + \dfrac{\phi(t)}{A}, & t \in (-\infty, 0], \\ \dfrac{\phi(0)}{A-1} + \dfrac{1}{A-1}\left[\displaystyle\int_0^\infty f(y_s)ds + \sum_{k=1}^{\infty} I_k(y(t_k))\right] & \\ + \displaystyle\int_0^t f(y_s)ds + \sum_{0<t_k<t} I_k(y(t_k)), & t \in [0,\infty), \end{cases} \tag{7}$$

*where* $\lim_{t\to\infty} y(t) = y_\infty$, *if and only if* $y$ *is a solution of the impulsive boundary value problem*

$$y'(t) = f(y_t); \ \ t \in [0,\infty), \ t \neq t_k, \quad k = 1, \ldots, \tag{8}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \ldots, \tag{9}$$

$$Ay(t) - y_\infty = \phi(t), \ t \in (-\infty, 0]. \tag{10}$$

**Proof.** Let $y$ be a solution of the impulsive integral equation (7), then for $t \in [0,\infty)$ and $t \neq t_k, \ k = 1, \ldots$ we have

$$y(t) = \frac{\phi(0)}{A-1} + \frac{1}{A-1}\left[\int_0^\infty f(y_s)ds + \sum_{k=1}^{\infty} I_k(y(t_k))\right] + \int_0^t f(y_s)ds + \sum_{0<t_k<t} I_k(y(t_k)).$$

Thus $y'(t) = f(y_t)$ for $t \in [0,\infty)$ and $t \neq t_k, \ k = 1, \ldots$.
From the definition of $y$ we can prove that

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \quad \text{for } k = 1, \ldots.$$

Finally we prove that $Ay(t) - y_\infty = \phi(t), \quad t \in (-\infty, 0]$. We have

$$\lim_{t\to\infty} y(t) = \frac{\phi(0)}{A-1} + \frac{A}{A-1}\left[\int_0^\infty f(y_s)ds + \sum_{k=1}^{\infty} I_k(y(t_k))\right]$$

and

$$y(t) = \frac{\phi(0)}{A(A-1)} + \frac{1}{A-1}\left[\int_0^\infty f(y_s)ds + \sum_{k=1}^{\infty} I_k(y(t_k))\right] + \frac{\phi(t)}{A}, \ \ t \in (-\infty, 0].$$

Hence

$$\begin{aligned}
Ay(t) - \lim_{t\to\infty} y(t) &= \frac{\phi(0)}{A-1} + \frac{A}{A-1}\left[\int_0^\infty f(y_s)ds + \sum_{k=1}^{\infty} I_k(y(t_k))\right] + \phi(t) \\
&\quad - \frac{\phi(0)}{A-1} - \frac{A}{A-1}\left[\int_0^\infty f(y_s)ds + \sum_{k=1}^{\infty} I_k(y(t_k))\right] \\
&= \phi(t), \ \ t \in (-\infty, 0]..
\end{aligned}$$

Let $y$ be a solution of the problem (8)–(10). Then

$$y'(t) = f(y_t) \ \text{ for } \ t \in [0, t_1].$$

An integration from 0 to $t$ (here $t \in (0, t_1]$) of both sides of the above equality yields

$$\int_0^t y'(s)ds = \int_0^t f(y_s)ds,$$

or

$$y(t) - y(0) = \int_0^t f(y_s)ds.$$

Thus for $t \in [0, t_1]$ we have

$$y(t) = y(0) + \int_0^t f(y_s)ds.$$

If $t \in (t_1, t_2]$, then we have

$$\begin{aligned}
\int_0^t y'(s)ds &= \int_0^t f(y_s s)ds, \\
\int_0^{t_1} y'(s)ds + \int_{t_1}^t y'(s)ds &= \int_0^t f(y_s)ds, \\
y(t_1) - y(0) + y(t) - y(t_1^+) &= \int_0^t f(s)ds, \\
y(t) - I_1(y(t_1)) &= y(0) + \int_0^t f(y_s)ds.
\end{aligned}$$

Hence for $t \in (t_1, t_2]$ we have

$$y(t) = y(0) + \int_0^t f(y_s)ds + I_1(y(t_1)).$$

Continue we obtain for $t \in [0, \infty)$ that

$$y(t) = y(0) + \int_0^t f(y_s)ds + \sum_{0<t_k<t} I_k(y(t_k)). \tag{11}$$

Since $\lim_{t\to\infty} y(t) = y_\infty$, we get

$$y_\infty \;=\; y(0) + \int_0^\infty f(y_s)ds + \sum_{k=1}^\infty I_k(y(t_k)).$$

Thus

$$y(0) = y_\infty - \int_0^\infty f(y_s)ds - \sum_{k=1}^\infty I_k(y(t_k)).$$

By (10) we have $y_\infty = Ay(0) - \phi(0)$ and hence

$$y(0) = \frac{\phi(0)}{A-1} + \frac{1}{A-1}\left[\int_0^\infty f(y_s)ds + \sum_{k=1}^\infty I_k(y(t_k))\right]. \tag{12}$$

We replace (12) in (11), we obtain

$$\begin{aligned} y(t) \;&=\; \frac{\phi(0)}{A-1} + \frac{1}{A-1}\left[\int_0^\infty f(s)ds + \sum_{k=1}^\infty I_k(y(t_k))\right] \\ &\quad + \int_0^t f(y_s)ds + \sum_{0<t_k<t} I_k(y(t_k)). \end{aligned}$$

From (10) and (12), we have

$$\begin{aligned} y(t) \;&=\; \frac{\phi(t)}{A} + \frac{1}{A}\left[y(0) + \int_0^\infty f(y_s)ds + \sum_{k=1}^\infty I_k(y(t_k))\right] \\ &=\; \frac{\phi(0)}{A-1} + \frac{1}{A-1}\left[\int_0^\infty f(y_s)ds + \sum_{k=1}^\infty I_k(y(t_k))\right] + \frac{\phi(t)}{A}, \;\; t \in (-\infty, 0]. \end{aligned}$$

**Theorem 2.1** *Let $f : J \times B \to \mathbb{R}^n$ be an $L^1$-Carathéodory function. Assume that:*

*(H1) There exist positive constants $c_k,\ k = 1, \ldots,$ such that*

$$|I_k(y)| \le c_k, \quad \textit{for all } y \in \mathbb{R}^n, \textit{ and } \sum_{k=1}^\infty c_k < \infty;$$

*(H2) There exists $p \in L^1([0, +\infty), \mathbb{R}_+)$ such that*

$$\|f(t, y)\| \le p(t) \quad \textit{for a.e. } t \in [0, \infty) \textit{ and each } y \in B.$$

*Then the boundary value problem (1)-(3) has at least one solution.*

**Proof.** Transform the problem (1)–(3) into a fixed point problem. Consider the operator $N : B_b \longrightarrow B_b$ defined by,

$$N(y)(t) = \begin{cases} \dfrac{\phi(0)}{A(A-1)} + \dfrac{1}{A-1}\left[\displaystyle\int_0^\infty f(s,y_s)ds + \sum_{k=1}^\infty I_k(y(t_k))\right] + \dfrac{\phi(t)}{A}, & t \in (-\infty, 0], \\ \dfrac{\phi(0)}{A-1} + \dfrac{1}{A-1}\left[\displaystyle\int_0^\infty f(s,y_s)ds + \sum_{k=1}^\infty I_k(y(t_k))\right] & \\ + \displaystyle\int_0^t f(s,y_s)ds + \sum_{0<t_k<t} I_k(y(t_k)), & t \in [0,\infty). \end{cases}$$

Let $x(\cdot) : (-\infty, +\infty) \to \mathbb{R}^n$ be the function defined by

$$x(t) = \begin{cases} \dfrac{\phi(0)}{A-1} + \dfrac{1}{A-1}\left[\displaystyle\int_0^\infty f(s,x_s)ds + \sum_{k=1}^\infty I_k(x(t_k))\right], & \text{if } t \in [0,\infty), \\ \dfrac{\phi(0)}{A(A-1)} + \dfrac{1}{A-1}\left[\displaystyle\int_0^\infty f(s,x_s)ds + \sum_{k=1}^\infty I_k(x(t_k))\right] + \dfrac{\phi(t)}{A}, & \text{if } t \in (-\infty, 0]. \end{cases}$$

Then $x_0 = \dfrac{\phi(0)}{A-1} + \dfrac{1}{A-1}\left[\displaystyle\int_0^\infty f(s,x_s)ds + \sum_{k=1}^\infty I_k(x(t_k))\right]$. For each $z \in C([0,\infty), \mathbb{R}^n)$ with $z(0) = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} z(t), & \text{if } t \in [0,\infty), \\ 0, & \text{if } t \in (-\infty, 0]. \end{cases}$$

If $y(\cdot)$ satisfies the integral equation,

$$y(t) = \frac{\phi(0)}{A-1} + \frac{1}{A-1}\left[\int_0^\infty f(s,y_s)ds + \sum_{k=1}^\infty I_k(y(t_k))\right] + \int_0^t f(s,y_s)ds + \sum_{0<t_k<t} I_k(y(t_k)),$$

we can decompose $y(\cdot)$ as $y(t) = \bar{z}(t) + x(t), 0 \le t < \infty$, which implies $y_t = \bar{z}_t + x_t$, for every $0 \le t < \infty$, and the function $z(\cdot)$ satisfies

$$z(t) = \int_0^t f(s, \bar{z}_s + x_s)ds + \sum_{0<t_k<t} I_k(\bar{z}(t_k) + x(t_k)). \tag{13}$$

Set

$$C_0 = \{z \in B_b : z(0) = 0\}.$$

Let the operator $P : C_0 \to C_0$ be defined by

$$(Pz)(t) = \begin{cases} 0 & t \le 0, \\ \displaystyle\int_0^t f(s, \bar{z}_s + x_s)ds + \sum_{0<t_k<t} I_k(\bar{z}(t_k) + x(t_k)), & t \in [0,\infty). \end{cases}$$

Obviously, that the operator $N$ has a fixed point is equivalent to $P$ has a fixed point, and so we turn to proving that $P$ has a fixed point. We shall use the Leray-Schauder alternative to prove that $P$ has a fixed point.

**Claim 1**: *$P$ is continuous.*

Let $\{z_n\}$ be a sequence such that $z_n \to z$ in $C_0$. Then

$$\begin{aligned}\|(Pz_n)(t) - (Pz)(t)\| &\leq \int_0^\infty \|f(s, \bar{z}_{n_s} + x_s) - f(s, \bar{z}_s + x_s)\|ds \\ &+ \sum_{k=1}^{\infty} \|I_k(z_{n_s} + x_s) - I_k(z_s + x_s)\|.\end{aligned}$$

On the other hand, for any $\varepsilon$, we can choose a positive integer $m$ such that $\sum\limits_{k=m+1}^{\infty} c_k < \varepsilon$. Then, choose an integer $n_0$ such that

$$\sum_{k=1}^{m} \|I_k(\bar{z}_n(t_k) + x(t_k)) - I_k(\bar{z}(t_k) + x(t_k))\| < \varepsilon, \ \forall n > n_0 \ (k = 1, \ldots, m). \tag{14}$$

From (14) we have

$$\sum_{k=1}^{\infty} \|I_k(\bar{z}_n(t_k) + x(t_k)) - I_k(\bar{z}(t_k) + x(t_k))\| < \varepsilon + \sum_{k=m+1}^{\infty} c_k < 2\varepsilon, \forall n > n_0.$$

Thus

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \|I_k(\bar{z}_n(t_k) + x(t_k)) - I_k(\bar{z}(t_k) + x(t_k))\| = 0. \tag{15}$$

Since $f$ is $L^1$-Carathéodory, we have

$$\begin{aligned}\|P(z_n) - P(z)\|_\infty &\leq \|f(\cdot, \bar{z}_{n_{(\cdot)}} + x_{(\cdot)}) - f(\cdot, \bar{z}_{(\cdot)} + x_{(\cdot)})\|_{L^1} \\ &\quad + \sum_{k=1}^{\infty} \|I_k(\bar{z}_n(t_k) + x(t_k)) - I_k(\bar{z}(t_k) + x(t_k))\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

**Claim 2**: *$P$ maps bounded sets into bounded sets in $C_0$.*

Indeed, it is enough to show that for any $q > 0$, there exists a positive constant $\ell$ such that for each $z \in \mathcal{B}_q = \{z \in C_0 : \|z\|_\infty \leq q\}$ one has $\|P(z)\|_\infty \leq \ell$. Let $z \in \mathcal{B}_q$. By (H2) we have for each $t \in [0, \infty)$

$$\|(Pz)(t)\| \leq \int_0^\infty p(s)ds + \sum_{k=1}^{\infty} c_k := \ell.$$

**Claim 3**: *$P$ maps bounded sets into equicontinuous sets of $C_0$.*

Let $l_1, l_2 \in [0, \infty),\ l_1 < l_2$ and $\mathcal{B}_q$ be a bounded set of $C_0$ as in Claim 2. Let $z \in \mathcal{B}_q$. Then for each $t \in [0, \infty)$ we have

$$\begin{aligned}\|(Pz)(l_2) - (Pz)(l_1)\| &\leq \int_{l_1}^{l_2} \|f(s, \bar{z}_s + x_s)\| ds + \sum_{0<t<l_2-l_1} \|I_k(\bar{z}(t_k) + x(t_k))\| \\ &\leq \int_{l_1}^{l_2} p(s) ds + \sum_{0<t<l_2-l_1} c_k.\end{aligned}$$

We see that $\|(Pz)(l_2) - (Pz)(l_1)\|$ tend to zero independently of $z \in \mathcal{B}_q$ as $l_2 - l_1 \to 0$. As a consequence of Claims 1 to 3 together with the Arzelá-Ascoli theorem we can conclude that $P : C_0 \longrightarrow C_0$ is continuous and completely continuous.

**Claim 4:** *A priori bounds on solutions.*

Let $z$ be a possible solution of the equation $z = \lambda P(z)$ and $z_0 = \lambda\phi$ for some $\lambda \in (0, 1)$. Then

$$\begin{aligned}\|z(t)\| &\leq \int_0^t \|f(s, \bar{z}_s + x_s)\| ds + \sum_{0<t_k<t} |I_k(\bar{z}(t_k) + x(t_k))| \\ &\leq \int_0^t p(s) ds + \sum_{k=1}^{\infty} c_k.\end{aligned}$$

This implies that for each $t \in [0, \infty)$

$$\|z\|_\infty \leq \|p\|_{L^1} + \sum_{k=1}^{\infty} c_k := K_1.$$

Set

$$U = \{z \in C_0 : \sup\{\|z(t)\| : 0 \leq t \leq b\} < K_1 + 1\}.$$

$P : \overline{U} \to C_0$ is continuous and completely continuous. From the choice of $U$, there is no $z \in \partial U$ such that $z = \lambda P(z)$, for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [8], we deduce that $P$ has a fixed point $z$ in $U$. Hence $N$ has a fixed point $y$ which is a solution to problem (1)–(3).

# 3 Neutral Functional Differential Equations

This section is concerned with the existence of solutions for boundary value problems for first order impulsive neutral functional differential equations with infinity delay (4)-(6).

**Theorem 3.1** *Let $f : J\times B \to \mathbb{R}^n$ be an $L^1$-Carathéodory function. Assume $(H1), (H2)$ and the condition:*

*(B1) The function $g$ is continuous and completely continuous and for any bounded set $Q \subseteq C((-\infty,\infty),\mathbb{R}^n)$ the set $\{t \to g(t,x_t) : x \in Q\}$ is equicontinuous in $C([0,\infty),\mathbb{R}^n)$ and there exist constants $c_1,\ c_2 \geq 0$ such that*

$$|g(t,u)| \leq c_1\|u\|_B + c_2, \quad t \in [0,\infty),\ u \in B \text{ and } c_1K_\infty < 1,$$

*where $K_\infty = \sup\{|K(t)| : t \in [0,\infty)\} < \infty$,*

*are satisfied. Assume also that $M_\infty = \sup\{|M(t)| : t \in [0,\infty)\} < \infty$, Then the boundary value problem (4)-(6) has at least one solution.*

**Proof.** In analogy to Theorem 2.1, we consider the operator $P^* : C_0 \to C_0$ defined by

$$(P^*z)(t) = \begin{cases} 0, & t \leq 0, \\ g(0,\phi) - g(t,\overline{z}_t + x_t) + \displaystyle\int_0^t f(s,\overline{z}_s + x_s)ds & \\ + \displaystyle\sum_{0<t_k<t} I_k(\bar{z}(t_k) + x(t_k)), & t \in [0,\infty). \end{cases}$$

In order to use the Leray-Schauder nonlinear alternative, we shall obtain a priori estimates for the solution of the integral impulsive equation

$$z(t) = \lambda\left[g(0,\phi) - g(t,\overline{z}_t + x_t) + \int_0^t f(s,\overline{z}_s + x_s)ds + \sum_{0<t_k<t} I_k(\bar{z}(t_k) + x(t_k))\right],$$

where $z_0 = \lambda\phi$ for some $\lambda \in (0,1)$. Then

$$\begin{aligned} \|z(t)\| &\leq \|g(0,\phi(0))\| + \|g(t,\bar{z}_t + x_t)\| + \int_0^t p(s)ds \\ &\quad + \sum_{0<t_k<t} \|I_k(\bar{z}(t_k) + x(t_k))\| \\ &\leq \|g(0,\phi)\| + c_1\|\bar{z}_t + x_t\|_B + c_2 + \int_0^t p(s)ds + \sum_{k=1}^{\infty} c_k. \end{aligned} \tag{16}$$

But

$$\begin{aligned} \|\bar{z}_s + x_s\|_B &\leq \|\bar{z}_s\|_B + \|x_s\|_B \\ &\leq K(s)\sup\{\|z(\tau)\| : 0 \leq \tau \leq s\} + K(s)\|z_0\|_B \\ &\quad +K(s)\sup\{\|x(\tau)\| : 0 \leq \tau \leq s\} + M(s)\|x_0\|_B. \end{aligned}$$

By the definition of $x$ we have

$$\sup\{\|x(\tau)\| : 0 \leq \tau \leq s\} \leq \frac{\|\phi\|_B}{A-1} + \frac{1}{A-1}\left[\int_0^\infty p(s)ds + \sum_{k=1}^{\infty} c_k\right] := K_*,$$

and

$$\|x_0\|_B \;\leq\; \frac{\|\phi\|_B}{A-1} + \frac{1}{A-1}\left[\int_0^{\infty} p(s)ds + \sum_{k=1}^{\infty} c_k\right].$$

Then

$$\begin{aligned}\|\bar{z}_s + x_s\|_B &\leq K(s)\sup\{\|z(\tau)\| : 0 \leq \tau \leq s\} + K(s)K_* + M(s)K_* \\ &\leq K_\infty \sup\{\|z(\tau)\| : 0 \leq \tau \leq s\} + K_\infty K_* + M_\infty K_*,\end{aligned}$$

Hence (16) implies that

$$\begin{aligned}\|z(t)\| \;\leq\;& \|g(0,\phi)\| + c_1 K_\infty K_* + c_1 M_\infty K_* + c_2 + \sum_{k=1}^{\infty} c_k \\ &+ c_1 K_\infty \sup\{\|z(s)\| : 0 \leq s \leq t\} + \int_0^{\infty} p(s)ds.\end{aligned}$$

Then

$$\begin{aligned}\|z\|_\infty \;\leq\;& \frac{1}{1 - c_1 K_\infty}\Big[\|g(0,\phi)\| + c_1 K_\infty K_* \\ +\;& c_1 M_\infty K_* + c_2 + \sum_{k=1}^{\infty} c_k + \int_0^{\infty} p(s)ds\Big] := K_{**}.\end{aligned}$$

Set

$$U_1 = \{z \in C_0 : \|z\|_\infty < K_{**} + 1\}.$$

From the choice of $U_1$, there is no $z \in \partial U_1$ such that $z = \lambda P^*(z)$, for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [8], we deduce that $P^*$ has a fixed point $z$ in $U_1$.

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