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On the behaviors of solutions to a functional differential equation of neutral type with multiple delays

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Abstract

We consider a linear functional differential equation of first order and neutral type with multiple constant delays and present a functional Lyapunov approach to square integrability and asymptotic analysis of solutions of the equation considered. Consequently explicit criteria are derived for square integrability and asymptotic analysis of solutions of the equation considered. A discussion of the result obtained and an illustrative example are given. The result given extends and improves some found in the literature.

1 Introduction

We know from the relevant literature that functional differential equations of neutral type have numerous applications in science, engineering, medicine

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and so on. For example, functional differential equations of neutral type are used as models of loss less transmission lines, partial element equivalent circuits (see Bellen et al. [3]), steam or water pipes, heat exchangers, (see Kolmanovskii and Myshkis [13]), control of constrained manipulators with delay measurements in mechanical engineering (see Niculescu and Brogliato [18]) and so on.

The traditional approaches to problems of qualitative analysis of time-varying functional differential equations of neutral type are based on the second method of Lyapunov and its variants. In general, it is not easy to tackle the problem of qualitative analysis of time-varying functional differential equations of neutral type. However, problems of qualitative properties of various forms of functional differential equations of neutral type have been studied intensively in the literature during the past decades, e.g. Agarwal and Grace [1], Altun and Tunç [2], Bellen et al. [3], Biçer and Tunç [4], Deng et al. [5], Gopalsamy [6], Gopalsamy et al. [7], Gopalsamy and Lung [8], Gopalsamy and Zhang [9], Gozen and Tunç [10], Gyori and Ladas [11], Hale [12], Kolmanovskii and Myshkis [13],] Krishnasamy and Balasubramaniam [14], Kwon and Park [15], Li and Jiang [16], Nam and Phat [17], Niculescu and Brogliato [18], Park [19], Rojsiraphisal and Niamsup [20], Ren and Li [21], Shanholt [22], Sun [23], Tang and Zou [24], Tunç ([25], [26], [27], [28],[29]), Zhang and Gopalsamy ([30], [31]).

In addition, we mention the following related results.

The asymptotic behaviors of solutions of the following differential equation with constant delay:

$$\frac{dx}{dt} + p(t)x + q(t)x(t-\sigma) = 0$$

or its different variants have been discussed in details by Gopalsamy [6], Gyori and Ladas [11] and Hale [12].

Later, in 2000, Agarwal and Grace [1] investigated the convergence of solutions to the following neutral differential equation with constant delay:

$$\frac{d}{dt}[x + c(t)x(t-\tau)] + p(t)x + q(t)x(t-\sigma) = 0. \tag{1.1}$$

In this paper, motivated by the works mentioned, we consider the following neutral differential equation with multiple constant delays:

$$\frac{d}{dt}[x + \sum_{i=1}^{n} c_i(t)x(t-\tau_i)] + \sum_{i=1}^{n} p_i(t)x + \sum_{i=1}^{n} q_i(t)x(t-\sigma_i) = 0, \tag{1.2}$$

where τ_i and $\sigma_i, i = 1, 2, ..., n$, are positive real numbers, $\sigma_i \geq \tau_i, c_i, p_i, q_i : [t_0, \infty) \to [0, \infty)$ are continuous functions such that $c_i(t), i = 1, 2, ...n$, are differentiable and have bounded derivatives. It can be seen that equation (1.2)includes equation (1.1) of [1] and the related ones in ([6], [11], [12]). The purpose of this article is to give new sufficient conditions, which guarantee that all solutions of equation (1.2) are square integrable and approach zero as $t \to \infty$. By defining a suitable Lyapunov functional, we prove a new theorem on the topic. Our result extends and improves the results of Agarwal and Grace [1, Theorem 1], Gopalsamy[6], Gyori and Ladas [11], Hale [12] and some recent results in the literature. Throughout the paper, x represents $x(t)$.

2 Square integrability and asymptotic stability

First, the following lemma is needed.
Lemma 1. Let $p \in [0, 1), \tau \in (0, \infty), t_0 \in R, x \in C[(t_0 - \tau, \infty), R^+]$ and assume that for every $\varepsilon > 0$ there exists a $t_\varepsilon \geq t_0$ such that:

$$x(t) \leq (p + \varepsilon)x(t - \tau) + \varepsilon \quad \text{for} \quad t \geq t_\varepsilon.$$

Then

$$\lim_{t \to \infty} x(t) = 0.$$

Proof. See Gyori and Ladas [11].
We now assume the following hypotheses in the main result of this paper.
Hypotheses
Suppose the following hypotheses hold.
(H1)

$$p_{1i} \leq p_i(t) \leq p_{2i}, q_{1i} \leq q_i(t) \leq q_{2i}, c_i(t) \leq c_{1i} < 1, |\acute{c}_i(t)| \leq c_{2i} \quad \text{for all} \quad t \geq t_0,$$

where $c_{1i}, c_{2i}, p_{1i}, p_{2i}$, and q_{2i} are nonnegative constants.
(H2)

$$p_{1i} + q_{1i} > (p_{2i} + q_{2i})(c_{1i} + iq_{2i}\sigma_i), i = 1, 2, ..., n.$$

(H3)

$$\sum_{i=1}^{n} p_{2i} \sum_{i=1}^{n} q_{2i}\sigma_i \geq \sum_{i=1}^{n} q_{2i} \sum_{i=1}^{n} p_{2i}\sigma_i.$$

Theorem 1. If hypotheses (H1) and (H2) hold, then every solution $x(t)$ of equation (1.2) is square integrable and satisfies $x(t) \to 0$ as $t \to \infty$.
Proof. Let

$$y(t) = x(t) + \sum_{i=1}^{n} c_i(t)x(t-\tau_i).$$

We define a new Lyapunov functional $V = V(t)$ by

$$\begin{aligned}
V(t) =&(y(t) - \sum_{i=1}^{n}\int_{t-\sigma_i}^{t} q_i(s+\sigma_i)x(s)ds)^2 \\
&+ \sum_{i=1}^{n}\int_{t-\sigma_i}^{t}[p_i(s+\sigma_i) + q_i(s+2\sigma_i)](\sum_{i=1}^{n}\int_{s}^{t} q_i(u+\sigma_i)x^2(u)du)ds \\
&+ \sum_{i=1}^{n}\int_{t-\tau_i}^{t}[p_i(s+\tau_i) + q_i(s+\sigma_i+\tau_i)]c_i(s+\tau_i)x^2(s)ds.
\end{aligned}$$

It is clear that the above Lyapunov functional is positive definite.

Next, we calculate the time derivative of the functional V along solutions of equation (1.2). Then

$$\begin{aligned}
\frac{dV(t)}{dt} =&2(y(t) - \sum_{i=1}^{n}\int_{t-\sigma_i}^{t} q_i(s+\sigma_i)x(s)ds) \\
&\times (y'(t) - \sum_{i=1}^{n} q_i(t+\sigma_i)x(t) + \sum_{i=1}^{n} q_i(t)x(t-\sigma_i)) \\
&- \sum_{i=1}^{n}[p_i(t) + q_i(t+\sigma_i)]\sum_{i=1}^{n}\int_{t-\sigma_i}^{t} q_i(u+\sigma_i)x^2(u)du \\
&+ \sum_{i=1}^{n}\int_{t-\sigma_i}^{t}[p_i(s+\sigma_i) + q_i(s+2\sigma_i)]\sum_{i=1}^{n} q_i(t+\sigma_i)x^2(t)ds \\
&+ \sum_{i=1}^{n}[p_i(t+\tau_i) + q_i(t+\sigma_i+\tau_i)]c_i(t+\tau_i)x^2(t) \\
&- \sum_{i=1}^{n}[p_i(t) + q_i(t+\sigma_i)]c_i(t)x^2(t-\tau_i).
\end{aligned}$$

Using equation (1.2) and some elementary estimates, it follows that

$$\begin{aligned}
\frac{dV(t)}{dt} =& 2(y - \sum_{i=1}^{n} \int_{t-\sigma_i}^{t} q_i(s+\sigma_i)x(s)ds) \\
& \times (\sum_{i=1}^{n} p_i(t)x + \sum_{i=1}^{n} q_i(t)x(t-\sigma_i) + \sum_{i=1}^{n} q_i(t+\sigma_i)x - \sum_{i=1}^{n} q_i(t)x(t-\sigma_i)) \\
& - \sum_{i=1}^{n}[p_i(t) + q_i(t+\sigma_i)] \sum_{i=1}^{n} \int_{t-\sigma_i}^{t} q_i(u+\sigma_i)x^2(u)du \\
& + \sum_{i=}^{n} \int_{t-\sigma_i}^{t} [p_i(s+\sigma_i) + q_i(s+2\sigma_i)] \sum_{i=1}^{n} q_i(t+\sigma_i)x^2(t)ds \\
& + \sum_{i=1}^{n}[p_i(t+\tau_i) + q_i(t+\sigma_i+\tau_i)]c_i(t+\tau_i)x^2 \\
& - \sum_{i=1}^{n}[p_i(t) + q_i(t+\sigma_i)]c_i(t)x^2(t-\tau_i) = -2(x + \sum_{i=1}^{n} c_i(t)x(t-\tau_i) \\
& - \sum_{i=1}^{n} \int_{t-\sigma_i}^{t} q_i(s+\sigma_i)x(s)ds) \times (\sum_{i=1}^{n}(p_i(t) + q_i(t+\sigma_i))x) \\
& - \sum_{i=1}^{n}[p_i(t) + q_i(t+\sigma_i)] \sum_{i=1}^{n} \int_{t-\sigma_i}^{t} q_i(u+\sigma_i)x^2(u)du \\
& + \sum_{i=}^{n} \int_{t-\sigma_i}^{t} [p_i(s+\sigma_i) + q_i(s+2\sigma_i)] \sum_{i=1}^{n} q_i(t+\sigma_i)x^2(t)ds \\
& + \sum_{i=1}^{n}[p_i(t+\tau_i) + q_i(t+\sigma_i+\tau_i)]c_i(t+\tau_i)x^2 - \sum_{i=1}^{n}[p_i(t) + q_i(t+\sigma_i)]c_i(t)x^2(t-\tau_i) \\
=& -2[\sum_{i=1}^{n} p_i(t) + q_i(t+\sigma_i)]x^2 - 2[\sum_{i=1}^{n} p_i(t) + q_i(t+\sigma_i)]x(t) \sum_{i=1}^{n} c_i(t)x(t-\tau_i) \\
& + 2[\sum_{i=1}^{n} p_i(t) + q_i(t+\sigma_i)]x(t) \sum_{i=1}^{n} \int_{t-\sigma_i}^{t} q_i(s+\sigma_i)x(s)ds \\
& - \sum_{i=1}^{n}[p_i(t) + q_i(t+\sigma_i)] \sum_{i=1}^{n} \int_{t-\sigma_i}^{t} q_i(u+\sigma_i)x^2(u)du \\
& + \sum_{i=1}^{n} \int_{t-\sigma_i}^{t} [p_i(s+\sigma_i) + q_i(s+2\sigma_i)] \sum_{i=1}^{n} q_i(t+\sigma_i)x^2(t)ds \\
& + \sum_{i=1}^{n}[p_i(t+\tau_i) + q_i(t+\sigma_i+\tau_i)]c_i(t+\tau_i)x^2 - \sum_{i=1}^{n}[p_i(t) + q_i(t+\sigma_i)]c_i(t)x^2(t-\tau_i).
\end{aligned}$$

Using the inequality

$$2x(t)x(s) \leq x^2(t) + x^2(s),$$

we have

$$\begin{aligned}
\frac{dV(t)}{dt} \leq & -2[\sum_{i=1}^{n} p_i(t) + q_i(t+\sigma_i)]x^2 \\
& -2[\sum_{i=1}^{n} p_i(t) + q_i(t+\sigma_i)]x(t)\sum_{i=1}^{n} c_i(t)x(t-\tau_i) \\
& +\sum_{i=1}^{n}[p_i(t) + q_i(t+\sigma_i)]x^2(t)\sum_{i=1}^{n}\int_{t-\sigma_i}^{t} q_i(s+\sigma_i)ds \\
& +\sum_{i=1}^{n}[p_i(t) + q_i(t+\sigma_i)]\sum_{i=1}^{n}\int_{t-\sigma_i}^{t} q_i(s+\sigma_i)x^2(s)ds \\
& -\sum_{i=1}^{n}[p_i(t) + q_i(t+\sigma_i)]\sum_{i=1}^{n}\int_{t-\sigma_i}^{t} q_i(u+\sigma_i)x^2(u)du \\
& +\sum_{i=1}^{n}\int_{t-\sigma_i}^{t}[p_i(s+\sigma_i) + q_i(s+2\sigma_i)]\sum_{i=1}^{n} q_i(t+\sigma_i)x^2(t)ds \\
& +\sum_{i=1}^{n}[p_i(t+\tau_i) + q_i(t+\sigma_i+\tau_i)]c_i(t+\tau_i)x^2 \\
& -\sum_{i=1}^{n}[p_i(t) + q_i(t+\sigma_i)]c_i(t)x^2(t-\tau_i).
\end{aligned}$$

so that

$$\begin{aligned}\frac{dV(t)}{dt} \leq & -2[\sum_{i=1}^{n} p_i(t) + q_i(t+\sigma_i)]x^2 \\ & -2[\sum_{i=1}^{n} p_i(t) + q_i(t+\sigma_i)]x(t)\sum_{i=1}^{n} c_i(t)x(t-\tau_i) \\ & +\sum_{i=1}^{n}[p_i(t) + q_i(t+\sigma_i]x^2(t)\sum_{i=1}^{n}\int_{t-\sigma_i}^{t} q_i(s+\sigma_i)ds \\ & +\sum_{i=1}^{n}\int_{t-\sigma_i}^{t}[p_i(s+\sigma_i) + q_i(s+2\sigma_i)]\sum_{i=1}^{n} q_i(t+\sigma_i)x^2(t)ds \\ & +\sum_{i=1}^{n}[p_i(t+\tau_i) + q_i(t+\sigma_i+\tau_i)]c_i(t+\tau_i)x^2(t) \\ & -\sum_{i=1}^{n}[p_i(t) + q_i(t+\sigma_i)]c_i(t)x^2(t-\tau_i).\end{aligned}$$

Using the equality

$$\alpha_i(t) = p_i(t) + q_i(t+\sigma_i), \quad i = 1, 2, ..., n,$$

it follows that

$$\begin{aligned}\frac{dV(t)}{dt} \leq & \{-2\sum_{i=1}^{n}\alpha_i(t) + \sum_{i=1}^{n}\alpha_i(t+\tau_i)c_i(t+\tau_i) \\ & +\sum_{i=1}^{n}\alpha_i(t)\sum_{i=1}^{n}\int_{t-\sigma_i}^{t} q_i(s+\sigma_i)ds + \sum_{i=1}^{n}\alpha_i(t)c_i(t)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n} q_i(t+\sigma_i)\sum_{i=1}^{n}\int_{t-\sigma_i}^{t}\alpha_i(s+\sigma_i)ds\}x^2\\
&-\sum_{i=1}^{n}\alpha_i(t)c_i(t)[x(t-\tau_i)+x]^2\\
&\le\{-2\sum_{i=1}^{n}\alpha_i(t)+\sum_{i=1}^{n}\alpha_i(t+\tau_i)]c_i(t+\tau_i)\\
&+\sum_{i=1}^{n}[\alpha_i(t)\sum_{i=1}^{n}\int_{t-\sigma_i}^{t}q_i(s+\sigma_i)ds+\sum_{i=1}^{n}\alpha_i(t)c_i(t)\\
&+\sum_{i=1}^{n} q_i(t+\sigma_i)\sum_{i=1}^{n}\int_{t-\sigma_i}^{t}\alpha_i(s+\sigma_i)ds\}x^2.
\end{aligned}
$$

By hypotheses $(H1), (H2)$ and $(H3)$, we obtain

$$
\begin{aligned}
\frac{dV(t)}{dt}\le&\sum_{i=1}^{n}\{-2p_{1i}-2q_{1i}+(p_{2i}+q_{2i})c_{1i}+(p_{2i}+q_{2i})\sum_{i=1}^{n}\int_{t-\sigma_i}^{t}q_{2i}ds+(p_{2i}+q_{2i})c_{1i}\\
&+q_{2i}\sum_{i=1}^{n}\int_{t-\sigma_i}^{t}(p_{2i}+q_{2i})ds\}x^2\\
=&\sum_{i=1}^{n}\{-2(p_{1i}+q_{1i})+2(p_{2i}+q_{2i})c_{1i}+2(p_{2i}+q_{2i})\sum_{i=1}^{n}q_{2i}\sigma_i\}x^2\\
=&-2\sum_{i=1}^{n}\{(p_{1i}+q_{1i})-(p_{2i}+q_{2i})(c_{1i}+\sum_{i=1}^{n}q_{2i}\sigma_i)\}x^2, t\ge T\ge t_0.
\end{aligned}
$$

An integration of both sides of the above inequality from T to t implies that

$$
V(t)+2\sum_{i=1}^{n}\{(p_{1i}+q_{1i})-(p_{2i}+q_{2i})(c_{1i}+\sum_{i=1}^{n}q_{2i}\sigma_i)\}\int_{T}^{t}x^2(s)ds\le V(T)<\infty.
$$

Let

$$\begin{aligned} p_{11}+p_{12}+...+p_{1n}&=K,\\ p_{21}+p_{22}+...+p_{2n}&=L,\\ q_{11}+q_{12}+...+q_{1n}&=M,\\ q_{21}+q_{22}+...+q_{2n}&=N,\\ c_{11}+c_{12}+...+c_{1n}&=R,\\ q_{21}\sigma_1+q_{22}\sigma_2+...+q_{2n}\sigma_n&=S. \end{aligned}$$

Clearly

$$V(t)+2[(K+M)-(L+N)(R+nS)]\int_T^t x^2(s)ds \le V(T)<\infty.$$

Hence, the functional $V(t)$ is bounded on $[T,\infty)$ and $x(t)\in L^2[T,\infty)$. That is; the solution $x(t)$ of equation (1.2) is square integrable. Since $V(t)$ is bounded on $[T,\infty)$, it can easily be seen that there exists a constant $K\geq 0$ such that

$$\begin{aligned} |x(t)| \le& K+\sum_{i=1}^n c_{1i}|x(t-\tau_i)|+\sum_{i=1}^n\int_{t-\sigma_i}^t q(s+\sigma_i)|x(s)|ds\\ \le& K+\sum_{i=1}^n c_{1i}|x(t-\tau_i)|+\sum_{i=1}^n q_{2i}\sigma_i \max_{t-\sigma_i\le s\le t}|x(s)|\\ \le& K+\sum_{i=1}^n (c_{1i}+q_{2i}\sigma_i)\max_{t-\sigma_i\le s\le t}|x(s)|. \end{aligned}$$

In view of the results of Agarwal and Grace [1, Theorem 1] and Lemma 1; that is, Gyori and Ladas [11, Lemma 1.5.3], we conclude that the solution $x(t)$ of equation (1.2) tends to zero when t goes to infinity. This finishes the proof of Theorem 1.

Example 1. We consider the following neutral differential equation with multiple constant delays:

$$\begin{aligned} &\frac{d}{dt}\left(x+\frac{1}{t}x(t-\tau_1)+\frac{1}{t^2}x(t-\tau_2)\right)+x\\ +&\left(\frac{1}{t}+\frac{1}{t^2}\right)x(t-\sigma_1)+\left(\frac{1}{t^2}+\frac{2}{t^3}\right)x(t-\sigma_2), t\geq 3, \end{aligned}$$

which is a special case of equation (1.2) and where τ_i and σ_i are positive real numbers with $\sigma_i \geq \tau_i, i = 1, 2$. Comparing the above equation with equation (1.2) and considering hypotheses $(H1)$ and $(H2)$, it follows that

$$\begin{aligned} c_1(t) &= \frac{1}{t} \leq \frac{1}{3} = c_{11} < 1, t \geq 3, \\ \acute{c}_1(t) &= \frac{1}{t^2}, |\acute{c}_1(t)| \leq \frac{1}{9} = c_{21}, t \geq 3, \\ c_2(t) &= \frac{1}{t^2} \leq \frac{1}{9} = c_{12} < 1, t \leq 3, \\ \acute{c}_2(t) &= -\frac{2}{t^3}, |\acute{c}_2(t)| \leq \frac{2}{27} = c_{22}, t \geq 3, \\ p_1(t) &= p_2(t) = 1, \end{aligned}$$

so that

$$\begin{aligned} p_{11} &= p_{21} = p_{12} = p_{22} = 1, \\ q_1(t) &= \frac{1}{t} + \frac{1}{t^2}, 0 = q_{11} \leq q_1(t) \leq \frac{4}{9} = q_{21}, t \geq 3, \\ q_2(t) &= \frac{1}{t^2} + \frac{2}{t^3}, 0 = q_{12} \leq q_2(t) \leq \frac{5}{27} = q_{22}, t \geq 3. \end{aligned}$$

Moreover, we have

$$p_{11} + q_{11} = 1 > \frac{13}{9}\Big(\frac{1}{3} + \frac{4}{9}\sigma_1\Big) = (p_{21} + q_{21})(c_{11} + q_{21}\sigma_1)$$

if $\sigma_1 < \frac{126}{156}$, and

$$p_{12} + q_{12} = 1 > \frac{32}{27}\Big(\frac{1}{9} + 2\frac{5}{27}\sigma_2\Big) = (p_{22} + q_{22})(c_{12} + 2q_{22}\sigma_2)$$

if $\sigma_2 < \frac{5697}{2880}$.
We conclude that

$$(p_{21} + p_{22})(q_{21}\sigma_1 + q_{22}\sigma_2) \geq (q_{21} + q_{22})(p_{21}\sigma_1 + p_{22}\sigma_2)$$

if $(1+1)(\frac{4}{9}\sigma_1 + \frac{5}{27}\sigma_2) \geq (\frac{4}{9} + \frac{5}{27})(\sigma_1 + \sigma_2)$.

Hence, all the conditions of Theorem 1 hold if the inequalities $\tau_1 \leq \sigma_1 < \frac{126}{156}$ and $\tau_2 \leq \sigma_2 < \frac{5697}{2880}$ hold. In addition, the above given equation has the solution $x(t) = e^{-t}$ so that $x(t) \to 0$ as $t \to \infty$. Obviously the solution $x(t) = e^{-t}$ is square integrable on the interval on $[T, \infty)$.

3 Discussion

We pay our attention to a linear functional differential equation of first order and neutral type with multiple constant delays. We establish new criteria for the square integrability and asymptotic stability of solutions to the equation considered by defining a new Lyapunov functional. Namely, by Theorem 1, we improve and extend the square integrable and asymptotic stability results that can be seen in the literature from very simple linear cases to a more general linear functional differential equation with multiple constant delays (see Agarwal and Grace [1], Gopalsamy [6], Gyori and Ladas [11] and Hale [12]). By means of the result obtained, we improve and extend the previous results in the literature. We give an example to show applicability of the result obtained and for illustrations.

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