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On the stability of solutions of neutral differential equations of first order

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Abstract

In this work, the existence and uniqueness of solutions, boundedness of solutions and stability of zero solution of a specific neutral differential equations of first order with multiple time delays are discussed by the fixed point method. An example is given to show the applicability of the results introduced. By this work, we aim to extend and improve some previous results can be found in the literature and to do a contribution to the subject and the literature.

1 Introduction

Retarded and neutral functional differential equations have many applications in science and engineering ([1-23]). Therefore, it can be seen from the relevant literature that various investigations dealt with qualitative properties of first order retarded and neutral functional differential and functional

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integro-differential equations such as stability, instability, asymptotically stability, convergence, boundedness, existence of periodic solutions and so on ([1-23] and the references of these sources).

One of the oldest methods used to investigate the qualitative properties of such functional differential equations is direct method of Lyapunov. This method primarily requires the construction of suitable Lyapunov function(s) or functional(s) as the case may be. However, the construction of suitable Lyapunov function(s) or functional(s) can be a difficult task. Because there is still no definitive technique to know how to construct meaningful functions or functionals in the general cases. Here, the detailed information related this fact is not introduced.

We now present some information about the related problem considered in this paper.

For example, the retarded functional differential equation

$$\frac{d}{dt}x(t) = -a(t)g(x(t-\tau(t)))$$

is of historical importance and has significant applications. The study of this equation go back to 1951 and it has attracted the attention of a large number of investigators (Smart [15]).

Jin and Luo [11] consider the following integro -differential equation

$$\frac{d}{dt}x(t) = -\int_{t-r(t)}^{t} a(t,s)x(s)ds.$$

The authors in [11] give some conditions ensuring that the zero solution of this equation is asymptotically stable by means of the fixed point theory.

Becker and Burton [6] study the scalar integro-differential equation of first order with variable delay

$$\frac{d}{dt}x(t) = -\int_{t-r(t)}^{t} a(t,s)g(x(s))ds.$$

They find conditions ensuring that the zero solution of this equation is asymptotically stable by changing to an exponentially weighted metric on a closed subset of C_{ψ}. They also parlay the methods for this equation into results for retarded equations

$$\frac{d}{dt}x(t) = -\int_{t-r(t)}^{t} a(t,s)g(s,x(s))ds.$$

and

$$\frac{d}{dt}x(t) = -a(t)g(x(t-\tau(t))).$$

Chen et al. [9] take into consideration the following two nonlinear neutral delay differential equations

$$\frac{d}{dt}x(t) - c(t)x(t-r(t))\frac{d}{dt}x(t-r(t)) = -a(t)x(t) + b(t)g(x(t-r(t)))$$

and

$$\frac{d}{dt}x(t) - c(t)x(t-r(t))\frac{d}{dt}x(t-r(t)) = -a(t)x(t) + \int_{t-r(t)}^{t} K(t,s)g(x(s))ds.$$

The authors present some criteria for asymptotic stability of the zero solutions of these equations by the fixed point method.

Ardjouni and Djoudi [4] consider the following nonlinear neutral integro-differential equation with variable delay, $\tau(t) \geq 0$,

$$\frac{d}{dt}x(t) = -\int_{t-\tau(t)}^{t} a(t,s)g(x(s))ds + \frac{d}{dt}G(t, x(t-\tau(t))).$$

They establish necessary and sufficient condition for the asymptotic stability of zero solution of this equation.

Soulhia et al. [14] consider the neutral integro-differential equation

$$\frac{d}{dt}x(t) = -\int_{t-\tau(t)}^{t} a(t,s)g(x(s))ds + c(t)\frac{d}{dt}x(t-\tau(t))$$

by using Banach's contraction mapping principle. They investigate the asymptotic stability and the result obtained is illustrated by an example.

Ardjouni and Djoudi [5] study the stability properties of the scalar nonlinear neutral differential equation

$$\frac{d}{dt}x(t) = -a(t)g(x(t-\tau(t))) + \frac{d}{dt}G(t, x(t-\tau(t)))$$

by Banach's contraction mapping principle. They establish an asymptotic stability theorem for the zero solution of this equation.

Motivated especially by the results of [5], the works mentioned and those can be found in the relevant literature, we consider the following non-linear neutral differential equations (NDEs) with two variable delays

$$\frac{d}{dt}x(t) = -\sum_{i=1}^{2} a_i(t)g_i(x(t-\tau_i(t))) + \frac{d}{dt}\sum_{i=1}^{2} G_i(t, x(t-\tau_i(t))) \qquad (1)$$

with the initial condition $x(t) = \psi(t)$ on $t \in [m(0), 0]$ such that $\psi \in C([m(0), 0], R)$, where $m(0) = \{\inf\{(t - \tau_1(t)), t - \tau_2(t))\}, t \geq 0\}$ Here $C(S_1, S_2)$ denotes the set of all continuous functions; $\varphi : S_1 \to S_2$ with the supremum norm $\|.\|$. Throughout this paper, we suppose that $a_i \in C(R^+, R^+)$, $(i = 1, 2)$, and the functions $\tau_i : R^+ \to R^+$ are differentiable with $t - \tau_i(t) \to \infty$ as $t \to \infty$. The functions $g_i(x)$ and $G_i(t, x)$ are continuous and satisfy Lipschitz condition in x. That is, there are positive constants E_i such that

$$|G_i(t, x) - G_i(t, y)| \leq E_i|x - y|, \tag{2}$$

We also assume that

$$g_i(0) = 0 \ \ and \ \ G_i(t, 0) = 0. \tag{3}$$

Let

$$E = \max\{E_1, E_2\}.$$

In addition, for each $\psi \in C([m(0), 0], R)$, a solution of NDE (1) through $(0, \psi)$ is a continuous function $x : [m(0), T) \to R$ for some positive constant at $T > 0$ such that x satisfies NDE (1) on $[0, T)$. We denote such a solution by $x(t) = x(t, 0, \psi)$. From the existence theory we know that for each $\psi \in C([m(0), 0], R)$ there exists a unique solution $x(t) = x(t, 0, \psi)$ of NDE (1) defined on $[0, \infty)$. We define $\|\psi\| := \max\{|\psi(t)| : m(0) \leq t \leq 0\}$. It should be noted that the absence of linear terms in NDE (1) makes it difficult to obtain a fixed point mapping for NDE (1). So, to make NDE (1) more tractable, we have to transform it. Therefore, it is needed to transform equation NDE (1) in a suitable form.

Definition. The zero solution of NDE (1) is said to be stable at $t = 0$ if, for every, $\varepsilon > 0$, there exists a $\delta > 0$ such that $\psi : [m(0), 0] \to (-\delta, \delta)$ implies that $|x(t)| < \varepsilon$ for $t \geq m(0)$.

2 Stability of solutions

First, we begin by transforming NDE (1) into an equivalent equation to which we apply the variation of parameters to define a fixed point mapping.

Theorem 1. Let $\psi : [m(0), 0] \to \Re$ be a given continuous initial function. If x is a solution of NDE (1) on an interval with on then is a solution of the integral equation

$$
\begin{aligned}
x(t) =&\{\psi(0) - \sum_{i=1}^{2} G_i(0, \psi(\tau_i(0))) - \sum_{i=1}^{2} \int_{-\tau_i(0)}^{0} H(s) g_i(\psi(s)) ds\} \exp(-\int_0^t H(v) ds) \\
&+ \sum_{i=1}^{2} G_i(t, x(t-\tau_i(t)) + \sum_{i=1}^{2} \int_{t-\tau_i(t)}^{t} H(s) g_i(x(s)) ds \\
&- \int_0^t \exp(-\int_0^t H(v) dv) H(s) \sum_{i=1}^{2} (\int_{s-\tau_i(s)}^{s} H(u) g_i(x(u)) du) ds \\
&+ \int_0^t \exp(-\int_0^t H(v) dv) \sum_{i=1}^{2} [-a_i(s) + H(s-\tau_i(s)(1-\tau_i'(s))] g_i(x(s-\tau_i(s))) \\
&- H(s) G_i(s, x(s-\tau_i(s))] ds \\
&+ \int_0^t \exp(-\int_0^t H(v) dv) H(s) [x(s) - \sum_{i=1}^{2} g_i(x(s)) ds, \qquad (4)
\end{aligned}
$$

where $H : [m(0), \infty) \to \Re$ is an arbitrary continuous function. Conversely, if a continuous function x is equal to ψ on $[m(0), 0]$ and is a solution of (4) on an interval $[0, \sigma)$, then x is a solution of NDE (1) on $[0, \tau]$.

Proof. Let x be a solution of NDE (1). We can state NDE (1) in the following equivalent form

$$
\begin{aligned}
\frac{d}{dt}\{x(t) - &\sum_{i=1}^{2} G_i(t, x(t-\tau_i(t)))\} = -H(t)[x(t) - \sum_{i=1}^{2} G_i(t, x(t-\tau_i(t))] \\
&+ \sum_{i=1}^{2} \frac{d}{dt} \int_{t-\tau_i(t)}^{t} H(s) g_i(x(s)) ds - \sum_{i=1}^{2} a_i(t) g_i(x(t-\tau_i(t))) \\
&+ \sum_{i=1}^{2} H(t-\tau_i(t))(1-\tau_i'(t)) g_i(x(t-\tau_i(t)) \\
&- \sum_{i=1}^{2} H(t) G_i(t, x(t-\tau_i(t))) + H(t)[x(t) - \sum_{i=1}^{2} g_i(x(t))]. \qquad (5)
\end{aligned}
$$

Multiplying both sides of equation (5) by the term $\exp(\int_0^t H(v)dv)$, it follows that

$$
\frac{d}{dt}\{x(t) - \sum_{i=1}^{2} G_i(t, x(t-\tau_i(t)))\} \exp(\int_0^t H(v) dv)
$$

$$
\begin{aligned}
=&- H(t)[x(t) - \sum_{i=1}^{2} G_i(t, x(t-\tau_i(t))\} \exp(\int_0^t H(v)dv) \\
&+ \exp(\int_0^t H(v)dv) \sum_{i=1}^{2} \frac{d}{dt} \int_{t-\tau_i(t)}^{t} H(s) g_i(x(s))ds \\
&- \exp(\int_0^t H(v)dv) \sum_{i=1}^{2} a_i(t) g_i(x(t-\tau_i(t))) \\
&+ \exp(\int_0^t H(v)dv) \sum_{i=1}^{2} H(t-\tau_i(t))(1-\tau_i'(t)) g_i(x(t-\tau_i(t)) \\
&- \exp(\int_0^t H(v)dv) \sum_{i=1}^{2} H(t) G_i(t, x(t-\tau_i(t))) \\
&+ \exp(\int_0^t H(v)dv) H(t)[x(t) - \sum_{i=1}^{2} g_i(x(t))].
\end{aligned}
$$

Integrating the last equality from 0 to any $t \in [0, T)$, we have

$$
\begin{aligned}
x(t) =&\{\psi(0) - \sum_{i=1}^{2} G_i(0, \psi(\tau_i(0))) \exp(-\int_0^t H(v)dv) \\
&+ \sum_{i=1}^{2} G_i(t, x(t-\tau_i(t)) \\
&+ \int_0^t \exp(-\int_s^t H(v)dv) \sum_{i=1}^{2} \frac{d}{ds} (\int_{s-\tau_i(s)}^{s} H(u) g_i(x(u))du)ds \\
&+ \int_0^t \exp(-\int_s^t H(v)dv) \sum_{i=1}^{2} [-a_i(s) + H(s-\tau_i(s))(1-\tau_i'(s))] g_i(x(s-\tau_i(s)))ds \\
&- \int_s^t \exp(-\int_s^t H(v)dv) H(s) G_i(s, x(s-\tau_i(s))]ds \\
&+ \int_0^t \exp(-\int_s^t H(v)dv)[x(s) - \sum_{i=1}^{2} g_i(x(s))ds.
\end{aligned}
$$

Applying the integration by parts to the third right term, we get

$$x(t) =\{\psi(0) - \sum_{i=1}^{2} G_i(0, \psi(\tau_i(0))) - \sum_{i=1}^{2} \int_{-\tau_i(0)}^{0} H(s)g_i(\psi(s))ds\} \exp(- \int_s^t H(v)dv)$$
$$+ \sum_{i=1}^{2} G_i(t, x(t - \tau_i(t)) + \sum_{i=1}^{2} \int_{t-\tau_i(t)}^{t} H(s)g_i(x(s))ds$$

$$- \int_0^t \exp(- \int_0^t H(v)dv)H(s) \sum_{i=1}^{2} (\int_{s-\tau_i(s)}^{s} H(u)g_i(x(u))du)ds$$
$$+ \int_0^t \exp(- \int_0^t H(v)ds) \sum_{i=1}^{2} [-a_i(s) + H(s - \tau_i(s)(1 - \tau_i'(s))]g_i(x(s - \tau_i(s)))$$
$$- H(s)G_i(s, x(s - \tau_i(s))]ds$$
$$+ \int_0^t \exp(- \int_0^t H(v)dv)H(s)[x(s) - \sum_{i=1}^{2} g_i(x(s))ds.$$

Thus, the first part of the proof is complete.

Conversely, we now assume that there exists a continuous function x such that it equals to ψ on $[m(0), 0]$ and satisfies integral equation (4) on an interval $[0, \sigma)$. Then , the function x is differentiable on $[0, \sigma)$. If we calculate the time derivative of this integral equation, then we can obtain NDE (1).

From equation (4) we shall derive a fixed point mapping P for NDE (1). But the challenge here is to choose a suitable metric space of functions on which the map P can be defined. Below a weighted metric on a specific space is defined. Let C be the set of real-valued bounded continuous functions on $[m(0), \infty)$ with the supremum norm $\|.\|$ that is, for $\varphi \in C$,

$$\|\varphi\| = \sup\{|\varphi(t)| : t \in [m(0), \infty)\}.$$

In other words, we carry out our investigations in the complete metric space (C, d), where d denotes the supremum metric

$$d(\varphi_1, \varphi_2) = \|\varphi_1 - \varphi_2\|$$

for $\varphi_1, \varphi_2 \in C$. For a given initial function $\psi : [m(0), 0] \to [-l, l]$, $l > 0$, $l \in \Re$, define the set

$$S_\psi = \{\varphi : [m(0), \infty) \to \Re | \varphi \in C, \varphi(t) = \psi(t) \;\; for \;\; t \in [m(0), 0], |\varphi(t)| \leq l\}.$$

Since S_ψ is a closed subset of C, then the metric space (S_ψ, d) is complete.

Theorem 2. Let $H : [m(0), \infty) \to \Re^+$ be a continuous function and define a mapping P on S_ψ as follows, $\forall \varphi \in S_\psi$,

$$(P\varphi)(t) = \psi(t) \quad if \quad [m(0), 0], \quad and$$

while for $t > 0$,

$$\begin{aligned}(P\varphi)(t) = &\{\psi(0) - \sum_{i=1}^{2} G_i(0, -\psi(\tau_i(0))) - \sum_{i=1}^{2} \int_{-\tau_i(0)}^{0} H(s) g_i(\psi(s)) ds\} \exp(-\int_0^t H(v) dv) \\ &+ \sum_{i=1}^{2} G_i(t, \varphi(t - \tau_i(t)) + \sum_{i=1}^{2} \int_{t-\tau_i(t)}^{t} H(s) g_i(\varphi(s)) ds \\ &- \int_0^t \exp(-\int_0^t H(v) dv) H(s) \sum_{i=1}^{2} (\int_{s-\tau_i(s)}^{s} H(u) g_i(\varphi(u)) du) ds\end{aligned}$$

$$\begin{aligned}&+ \int_0^t \exp(-\int_0^t H(v) dv) \sum_{i=1}^{2} [-a_i(s) + H(s - \tau_i(s)(1 - \tau_i'(s))] g_i(\varphi(s - \tau_i(s))) \\ &- H(s) G_i(s, \varphi(s - \tau_i(s))] ds \\ &+ \int_0^t \exp(-\int_0^t H(v) dv) H(s) [\varphi(s) - \sum_{i=1}^{2} g_i(\varphi(s))] ds. \qquad (6)\end{aligned}$$

Suppose that (2) holds and the following conditions are satisfied: There exist a constant $l > 0$ such that g_i functions satisfy Lipschitz condition on $[-l, l]$ and let L be the Lipschitz constant for both $g_i(x)$ and $x - g_i(x)$ on $[-l, l]$ and $H(t) \geq 0$ for $t \geq m(0)$. Assume further

$$k > 9, E = \max\{E_1, E_2\} \quad and \quad kE \leq 1. \qquad (7)$$

where E_i is given by (2). Then there is a metric for d_h for S_ψ such that (S_ψ, d_h) is complete and P is a contraction on (S_ψ, d_h) if P maps S_ψ into itself.

Proof. It is clear that $P\varphi$ is continuous. Now, for $t \in [m(0), \infty]$ and a constant $k > 9$ define

$$h_i(t) = kL \sum_{i=1}^{2} \int_0^t [H(v) + w_i(v)] dv,$$

where

$$w_i(t) = \begin{cases} 0, & v \in [m(0), 0] \\ | - a_i(v) + H(v - \tau_i(v))(1 - \tau_i'(v)))| + \frac{E_i H(v)}{L} & v \in [0, \infty), \end{cases} \tag{8}$$

$i = 1, 2.$

Let S be the space of all continuous functions $\varphi : [m(0), \infty) \to \Re$ such that

$$|\varphi|_h := \sup\{|\varphi(t)| \exp(-\sum_{i=1}^{2} h_i(t)) : t \in [m(0), \infty)\} < \infty.$$

Then $(S_\psi, |.|_h)$ is a Banach space, which can be checked by Cauchy criterion for uniform convergence. Thus, (S_ψ, d_h) is a complete metric space, where d_h denotes the induced metric $d_h(\varphi, \eta) = |\varphi - \eta|_h$ for $\varphi, \eta \in S$. Being closed in S with this metric, the space $(S_\psi, |.|_h)$ is also complete.

Suppose that $P : S_\psi \to S_\psi$. We now need to show that P defined by (6) is a contraction. Toward this end, let $\varphi, \eta \in S_\psi$, since the functions g_i and G_i satisfy Lipschitz condition on $[-l, l]$ then it follows that

$$\begin{aligned} &|(P\varphi)(t) - (P\eta)(t)| \exp(-\sum_{i=1}^{2} h_i(t)) \\ &\qquad \le \sum_{i=1}^{2} E_i |\varphi(t - \tau_i(t)) - \eta(t - \eta_i(t))| \exp(\sum_{i=1}^{2} -h_i(t) - h_i(t - \tau_i(t)) + h_i(t - \tau_i(t)))) \\ &\qquad\quad + \sum_{i=1}^{2} \int_{t-\tau_i(t)}^{t} H(s) L |\varphi(s) - \eta(s)| \exp(\sum_{i=1}^{2} -h_i(t) + h_i(s) - h_i(s)) ds \end{aligned}$$

$$
+\int_0^t \exp(-\int_0^t H(v)dv)H(s)\sum_{i=1}^{2}(\int_{s-\tau_i(s)}^{s} H(u)L|\varphi(u)-\eta(u)|
$$

$$
\times\exp(-\sum_{i=1}^{2} h_i(t)+h_i(u)-h_i(u))duds
$$

$$
+\int_0^t \exp(-\int_0^t H(v)dv)\sum_{i=1}^{2}\{L|-a_i(s)+H(s-\tau_i(s))(1-\tau_i'(s)|+E_iH(s)\}
$$

$$
\times|\varphi(s-\tau_i(s))-\eta(s-\tau_i(s))|\exp(\sum_{i=1}^{2}-h_i(t)+h_i(s-\tau_i(s))-h_i(s-\tau_i(s)))ds
$$

$$
+\int_0^t \exp(-\int_0^t H(v)dv)H(s)L|\varphi(s)-\sum_{i=1}^{2}\eta(s)|\exp(\sum_{i=1}^{2}-h_i(t)+h_i(s)-h_i(s))ds \tag{9}
$$

for $t>0$. Let us denote all terms on the right side of inequality (9), respectively, by I_n, $n=1,2,...,5$.

For $t-\tau_i(t)\le s\le v\le t$, $i=1,2$, we have

$$
\begin{aligned}
\sum_{i=1}^{2}-h_i(t)+\sum_{i=1}^{2}h_i(s) &= -kL\sum_{i=1}^{2}\int_0^t[H(v)+w_i(v)]dv+kL\sum_{i=1}^{2}\int_0^s[H(v)+w_i(v)]dv\\
&= -kL\sum_{i=1}^{2}\int_s^t[H(v)+w_i(v)]dv\\
&\le -kL\int_s^t H(v)dv.
\end{aligned}
$$

Then, we observe

$$
I_2\le\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}\exp(-kL\int_s^t H(v)dv)H(s)L|\varphi(s)-\eta(s)|\exp(-\sum_{i=1}^{2}h_i(s))ds.
$$

For $s-\tau_i(s)\le u\le v\le s$, $i=1,2$, it follows

$$
\begin{aligned}
\sum_{i=1}^{2}-h_i(t)+\sum_{i=1}^{2}h_i(u) &= -kL\sum_{i=1}^{2}\int_0^t[H(v)+w_i(v)]dv+kL\sum_{i=1}^{2}\int_0^s[H(v)+w_i(v)]dv,\\
&= -kL\sum_{i=1}^{2}\int_s^t[H(v)+w_i(v)]dv\\
&\le -kL\int_s^t H(v)dv.
\end{aligned}
$$

Hence, we have

$$I_3 \leq \int_0^t \exp(-\int_s^t H(v)dv)H(s) \\ \times \sum_{i=1}^{2} \int_{s-\tau_i(s)}^{s} \exp(-kL\int_u^s H(u)dv)H(u)L|\varphi(u)-\eta(u)|\exp(-\sum_{i=1}^{2} h_i(s))duds.$$

Similarly, by expression (8) and $s-\tau_i(s) \leq t$, $i = 1, 2$, we get

$$\begin{aligned}\sum_{i=1}^{2} -h_i(t) + \sum_{i=1}^{2} h_i(s-\tau_i(s)) &= -kL\sum_{i=1}^{2}\int_0^t [H(v)+w_i(v)]dv + kL\sum_{i=1}^{2}\int_0^{s-\tau_i(s)} [H(v)+w_i(v)]dv, \\ &= -kL\sum_{i=1}^{2}\int_{s-\tau_i(s)}^{t} [H(v)+w_i(v)]dv \\ &\leq -kL\int_s^t H(v)dv.\end{aligned}$$

Then

$$I_4 \leq \sum_{i=1}^{2}\int_0^t \exp(-kL\sum_{i=1}^{2}\int_s^t w_i(v)dv)w_i(s) \\ \times |\varphi(s-\tau_i(s)-\eta(s-\tau_i(s))|\exp(\sum_{i=1}^{2} -h_i(s-\tau_i(s))ds.$$

In view of the above discussion, it can be obtained from (9) that

$$
\begin{aligned}
&|(P\varphi)(t)-(P\eta)(t)|\exp(-\sum_{i=1}^{2}h_i(t))\\
&\qquad\le \sum_{i=1}^{2}(\exp(-kL\int_{t-\tau_i(t)}^{t}\sum_{i=1}^{2}([H(v)+w_i(v)]dv)\\
&\qquad\times E_i|\varphi(t-\tau_i(t))-\eta(t-\tau_i(t))|\exp(\sum_{i=1}^{2}-h_i(t-\tau_i(t))\\
&\qquad+\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t}\exp(-kL\int_u^t H(v)dv)H(s)L|\varphi(s)-\eta(s)|\exp(\sum_{i=1}^{2}-h_i(s))ds\\
&\qquad+\int_0^t\exp(-\int_s^t H(v)dv)H(s)\sum_{i=1}^{2}\int_{s-\tau_i(s)}^{s}\exp(-kL\int_u^s H(v)dv)H(u)\\
&\qquad\times L|\varphi(u)-\eta(u)|\exp(-\sum_{i=1}^{2}h_i(u))duds.
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{2}\int_0^t\exp(-kL\sum_{i=1}^{2}\int_s^t w_i(v)dv)w_i(s)|\varphi(s-\tau_i(s)-\eta(s-\tau_i(s))|\exp(-\sum_{i=1}^{2}h_i(s-\tau_i(s))ds\\
&+\int_0^t\exp(-(kL+1)\int_s^t H(v)dv)H(s)L|\varphi(s)-\eta(s)|\exp(\sum_{i=1}^{2}-h_i(s))ds.
\end{aligned}
$$

Consequently, by using (7), we obtain

$$|(P\varphi)(t)-(P\eta)(t)|\exp(\sum_{i=1}^{2}-h_i(t))\le[\frac{2}{k}+(\frac{6}{kl}+\frac{1}{kL+1})L]|\varphi-\eta|_h$$

for $t>0$. We observe that for $P\varphi$ and $P\eta$ agree on $[m(0),0]$ and the last inequality holds for all $t\ge m(0)$. Therefore, we have

$$|P\varphi-P\eta|\le\frac{9}{k}|\varphi-\eta|_h,$$

Since $k>9$, we can conclude that P is a contraction on $(S_\psi,|.|_h)$.

Finally, we choose $\|\psi\|$ sufficiently small to prove the existence and uniqueness of solutions by showing that $P:S_\psi\to S_\psi$.

Theorem 3. Suppose the functions g_i, G_i and H satisfy the assumptions of Theorem 2 and (2), (3) and (7). In addition, we assume suppose the following assumptions hold:

(A1) g_i, $(i = 1, 2)$, are odd and strictly increasing on $[-l, l]$, $x - g_i(x)$ are non-decreasing on $[0, l]$.

(A2) There exists an $\alpha \in (0, 1)$ such that

$$4l \sum_{i=1}^{2} E_i + g_i(l)(\sum_{i=1}^{2} \int_{t-\tau_i(t)}^{t} H(s)ds) + \int_0^t \exp(-\int_s^t H(v)dv)H(s)(\int_{s-\tau_i(s)}^{s} H(u)du)ds$$
$$+ \int_0^t \exp(-\int_s^t H(v)dv) \sum_{i=1}^{2} |-a_i(s) + H(s - \tau_i(s)(1 - \tau_i'(s))|ds)$$
$$\leq \alpha \sum_{i=1}^{2} g_i(l), \forall t \geq 0.$$

Then there exists a constant $\delta \in (0, l)$ such that for each initial continuous function $\psi : [m(0), 0] \to (-\delta, \delta)$ there is a unique continuous function $x : [m(0), \infty) \to \Re$ with $x = \psi$ on $[m(0), 0]$, which is a solution of NDE (1) on $[0, \infty)$. Moreover, x is bounded by l on $[m(0), \infty)$. Furthermore, the zero solution of NDE (1) is stable at $t = 0$.

Proof. Since the functions g_i are odd and satisfy the Lipschitz condition on $[-l, l]$, $g_i(0) = 0$ and g_i are uniformly continuous on $[-l, l]$. Therefore, we can choose a δ that satisfies

$$\delta(l + \sum_{i=1}^{2} E_i) + \sum_{i=1}^{2} g_i(\delta) \int_{-\tau_i(0)}^{0} H(s)ds \leq (1 - \alpha) \sum_{=1}^{2} g_i(l) \tag{10}$$

Let $\psi : [m(0), 0] \to (-\delta, \delta)$ be a continuous function. Note that (10) implies $\delta < l$ since $g_i(l) \leq l$ by condition (A1). Thus, $|\psi(t)| \leq l$ for $m(0) \leq t \leq 0$. Now we show that for such a ψ the mapping $P : S_\psi \to S_\psi$. Indeed, consider

(6). For an arbitrary $\varphi \in S_\psi$, it follows from (2), (3), (A1) and (A2) that

$$\begin{aligned}|(P\varphi)(t)| \leq & \delta(1+\sum_{i=1}^{2} E_i) + \sum_{i=1}^{2} g_i(\delta)\int_{-\tau_i(0)}^{0} H(s)ds + l\sum_{i=1}^{2} E_i \\ & + \sum_{i=1}^{2} g_i(l)\int_{t-\tau_i(t)}^{t} H(s)ds \\ & + \sum_{i=1}^{2} g_i(l)\int_0^t \exp(-\int_s^t H(v)dv)H(s)(\int_{s-\tau_i(s)}^{s} H(u)du)ds \\ & + \int_0^t \exp(-\int_s^t H(v)dv)\{\sum_{i=1}^{2} g_i(l)|-a_i(s)+H(s-\tau_i(s)(1-\tau_i'(s))| \\ & + IE_iH(s)|\}ds + \sum_{i=1}^{2}(l-g_i(l))\int_0^t \exp(-\int_s^t H(v)dv)H(s)ds, \forall t>0.\end{aligned}$$

By applying (A2) and (10), we see that

$$\begin{aligned}|(P\varphi)(t)| \leq & \delta(1+\sum_{i=1}^{2} E_i) + \sum_{i=1}^{2} g_i(\delta)\int_{-\tau_i(0)}^{0} H(s)ds + \alpha\sum_{i=1}^{2} g_i(l) + l - \sum_{i=1}^{2} g_i(l) \\ & \leq (1-\alpha)\sum_{i=1}^{2} g_i(l) + \alpha\sum_{i=1}^{2} g_i(l) + l - \sum_{i=1}^{2} g_i(l) \\ & \leq g_1(l)+g_2(l)-\alpha g_1(l)-\alpha g_2(l)+\alpha g_1(l)+\alpha g_2(l)-g_1(l)-g_2(l)=l.\end{aligned}$$

Hence, $|(P\varphi)(t)| \leq l$ for $t \in [m(0),\infty)$ because $|(P\varphi)(t)| = |\psi(t)| \leq l$ for $t \in [m(0),0]$. Therefore, $P\varphi \in S_\psi$. By Theorem 2, P is a contraction on the complete metric space (S_ψ, d_h). Then P has a unique fixed point $x \in S_\psi$. Thus, $|x(t)| \leq l$ for $\forall t \in m(0)$ and is a solution of NDE (1) on $[0,\infty)$ by Theorem 1. Hence x is the only continuous function satisfying NDE (1) such that $x = \psi$ on $t \in [m(0),0]$. To prove the stability at $t = 0$, let $\varepsilon > 0$ be given and choose $r > 0$ such that $r < \min\{\varepsilon, l\}$. Replacing l with r beginning with (10), we can conclude that there is a constant $\delta > 0$ such that $\|\psi\| < \delta$ implies that unique continuous solution x agreeing on $[m(0),0]$ with ψ, satisfies $|x(t)| \leq r < \varepsilon$ for all $t \geq 0$.

Example. Consider the following nonlinear neutral differential equation with multiple variable delays $\tau_1(t), \tau_1(t)$:

$$\frac{d}{dt}x(t) = -\sum_{i=1}^{2} a_i(t)g_i(x(t-\tau_i(t))) + \frac{d}{dt}\sum_{i=1}^{2} G_i(t, x(t-\tau_i(t))), \qquad (11)$$

where $\tau_1(t) = 0.053t,\ \tau_2(t) = 0.947t,\ \alpha_1(t) = 0.701/(0,947t+1),\ \alpha_1(t) = 0.697/(0,873t+1),\ g_1(t) = \sin x,\ g_2(t) = \sin x,\ G_1(t) = 0,069\sin(x/3)$, $G_2(t) = 0,057\sin(x/3)$. We show that this equation satisfies the conditions of Theorems 1-3. Let $l = \pi/3$, $k = 10$ and $H(t) = 1/(t+1)$. We can see that $E_1(t) = 0.023$, $E_2(t) = 0,019$, $g_1(0) = 0$, $g_2(0) = 0$, $G_1(t,0) = 0$ and $G_2(t,0) = 0$. Clearly, the functions g_1 and g_2 are odd and strictly increasing on $[-\pi/3, \pi/3]$, $x - g_1(x)$ and $x - g_2(x)$ are non-decreasing on $[0, \pi/3]$.

It is notable that

$$\sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t} H(s)ds = \int_{0.947t}^{t}\frac{1}{s+1}ds + \int_{0.837t}^{t}\frac{1}{1+s}ds$$

$$= \ln(\frac{t+1}{0.947t+1}) + \ln(\frac{t+1}{0.837t+1})$$
$$< 0.191,$$

$$\int_0^t \exp(-\int_s^t H(v)dv)H(s)\sum_{i=1}^{2}(\int_{s-\tau_i(s)}^{s} H(u)du)ds < 0.191$$

and

$$\int_0^t e^{-\int_s^t H(v)dv}(\sum_{i=1}^{2}|-a_i(s) + H(s-\tau_i(s)(1-\tau_i'(s))|ds$$
$$= \int_0^t e^{-\int_s^t \frac{1}{v+1}dv}[(-\frac{0.701}{0.947s+1} + \frac{0.947}{0.9407s+1}) + (-\frac{0.697}{0.873s+1} + \frac{0.873}{0.873s+1})]$$
$$< (\frac{0.246}{0.947} + \frac{0.176}{0.873})\int_0^t e^{\int_s^t \frac{1}{v+1}dv}\frac{1}{s+1}ds < 0.463.$$

Then, we can conclude t that all the conditions of Theorems 1-3 hold provided that

$$\alpha = [(\pi/3)/(\sqrt{3})](0.168) + 0.194 + 0.194 + 0.463 \cong ..953 < 1.$$

Thus the zero solution of equation (11) is stable.

3 Conclusion

Benefited from the fixed point method, the qualitative properties such as the existence and uniqueness of solutions, boundedness of solutions and stability of zero solution of a certain neutral differential equation of first order with multiple time delays are investigated. To reach the purpose of the paper, three theorems are proved related to the subject of the paper. We introduce an example to verify the applicability of the results established. The obtained results include and improve that in [4] and do a contribution to the topic and literature.

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