International Journal of Mathematics and
Computer Science, **16**(2021), no. 1, 239–251

Pseudo KUS-algebras

Deena Al-Kadi

Department of Mathematics and Statistics
Faculty of Science
Taif University
Taif, Saudi Arabia

email: dak12le@hotmail.co.uk

(Received June 23, 2020, Accepted July 20, 2020)

Abstract

This paper introduces a generalization of KUS-algebras named pseudo KUS-algebras. In pseudo KUS-algebras, we characterize the notion of minimal elements and we consider the center and the G-part and investigate further properties.

1 Introduction

The algebraic structure of KUS-algebras were given in [1] and KUS-algebras homomorphisms and ideals were introduced. It was shown that such algebras were related to KU-algebras and some abelian groups. The reader can find more about KU-algebras in [2].

In [3, 4], it was shown that abstract algebras fall into two classes: BCK-algebras and BCI-algebras. The pseudo notion for these algebras has already been investigated by several authors; for example, Georgescu and Iorgulescu extended BCK-algebras to pseudo BCK-algebras in [5]. The notion of pseudo BCI-algebras were introduced by Dudek and Jun and some axioms for pseudo BCI-algebras to be pseudo BCK-algebras were given [6]. The pseudo BCK/BCI-algebras were also considered in [7] and properties of minimal elements were studied. The main subject in [8] was minimal elements (or atoms as called in that paper) and the connection between minimal

Key words and phrases: Pseudo, KUS-algebras, KU-algebras.
AMS (MOS) Subject Classifications: 06F35; 03G25, 03G27
ISSN 1814-0432, 2021, http://ijmcs.future-in-tech.net

elements and ideals were investigated. The pseudo notion has been studied for other algebras as well as BCH-algebras (see [9, 10]) in which the relation between pseudo BCH-algebras and BCI-algebras was given. In [11], the notions of pseudo BE-algebras, pseudo upper sets and pseudo filters were introduced and it was shown that the union of pseudo upper sets is a pseudo filter. In [12, 13], pseudo BL-algebras and pseudo d-algebras were extended to BL-algebras and d-algebras.

The main purpose of this paper is to construct and study new algebras called pseudo KUS-algebras ($PKUS$-algebras for short). In Section 2, we give the definition of KUS-algebras and KU-algebras. In Section 3, we introduce $PKUS$-algebras and give their characterization in Theorem 3.10. Also, we extend many properties of KUS-algebras to $PKUS$-algebras and discuss the relation between KU/KUS-algebras and $PKUS$-algebras. In Section 4, we provide a characterizations of minimal elements in $PKUS$-algebras through Theorem 4.3. This is motivated by the work done in [7, 8, 9]. Then we introduce the center of a $PKUS$-algebra X which is the set of all minimal elements in X. Finally, in Section 5, we investigate a subclass of $PKUS$-algebras X, named the G-part of X and we identify its properties.

2 The algebras KUS and KU

From [1], we recall definitions and propositions of KUS-algebras and KU-algebras that are needed in this paper.

Definition 2.1. *A triple* $(X, *, 0)$, *where* X *is a nonempty set,* $*$ *is a binary operation and* 0 *is an element of* X, *is called a* KUS*-algebra of type* $(2, 0)$ *if it satisfies the following axioms, for all* $x, y, z \in X$:

(1) $(x * y) * (x * z) = y * z$,
(2) $0 * y = y$,
(3) $y * y = 0$,
(4) $y * (x * z) = x * (y * z)$.

Proposition 2.2. *In a* KUS*-algebra* X, *the following properties are held, for all* $x, y, z \in X$:

(k_1) $x * z = 0$ *and* $z * x = 0 \Rightarrow x = z$,
(k_2) $x * (z * x) = z * 0$,
(k_3) $x * z = 0 \Rightarrow x * 0 = z * 0$,
(k_4) $(x * z) * 0 = z * x$,
(k_5) $x * 0 = 0 \Rightarrow x = 0$,

(k_6) $x = 0 * (0 * x)$,
(k_7) $0 * (x * z) = (0 * x) * (0 * z)$,
(k_8) $x * y = z * y \Rightarrow x * 0 = z * 0$.

Definition 2.3. *A KU-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$, where X is a nonempty set, $*$ is a single binary operation and 0 is an element of X satisfying the following axioms, for every $x, y, z \in X$:*
($KU1$) $(x * z) * [(z * y) * (x * y)] = 0$,
($KU2$) $0 * y = y$,
($KU3$) $y * 0 = 0$,
($KU4$) $x * z = 0$ *and* $z * x = 0 \Rightarrow x = z$.

Note that if $(x * y) * (x * z) = y * z$ in a KU-algebra, then it is a KUS-algebra.

3 Pseudo KUS-algebras

In this section, we introduce a generalization of KUS-algebras called pseudo KUS-algebras, denoted by $PKUS$ for short. We will use this notation in the rest of the paper. We define a partial order relation on $PKUS$-algebras and study some properties. Then we give a characterization of $PKUS$-algebras and investigate the relation between $PKUS$-algebras and pseudo KU-algebras.

Definition 3.1. *A structure $(X, *, \diamond, 0)$, where X is a nonempty set, $*, \diamond$ are binary operations on X and 0 is an element of X, is a $PKUS$-algebra of type $(2, 2, 0)$ if for all $x, y, z \in X$:*
($PKUS1$) $(x * y) \diamond (x * z) = y * z$,
$(x \diamond y) * (x \diamond z) = y \diamond z$,
($PKUS2$) $0 * y = y$ *and* $0 \diamond y = y$,
($PKUS3$) $y * y = 0$ *and* $y \diamond y = 0$,
($PKUS4$) $y * (x \diamond z) = x \diamond (y * z)$ *and* $y \diamond (x * z) = x * (y \diamond z)$.

Remark 3.2. *If $(X, *, 0)$ and $(X, \diamond, 0)$ are both KUS-algebras, then it is not necessary for $(X, *, \diamond, 0)$ to be a $PKUS$-algebra as shown in the following example.*

Example 3.3. *For $\{0, 1, 2, 3\} := X$, where $*$ and $\diamond$ are given by Table 1 and Table 2. By direct calculations, it is easy to check that $(X, *, 0)$ and $(X \diamond, 0)$ are both KUS-algebras whereas $(X, *, \diamond, 0)$ is not a $PKUS$-algebra since the axiom $PKUS1$ is not valid as $(3 * 2) \diamond (3 * 1) = 1 \diamond 2 = 1 \neq 2 * 1 = 3$.*

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Table 1:

$\diamond$	0	1	2	3
0	0	1	2	3
1	3	0	1	2
2	2	3	0	1
3	1	2	3	0

Table 2:

Remark 3.4. *$(X, \diamond, *, 0)$ is a $PKUS$-algebra whenever $(X, *, \diamond, 0)$ is a $PKUS$-algebra.*

Note that every KUS-algebra is a $PKUS$-algebra as if $x \diamond y := x * y$, in a KUS-algebra $(X, *, 0)$, then we get a $PKUS$-algebra $(X, *, \diamond, 0)$. If $x * y \neq x \diamond y$, then we say that the $PKUS$-algebra is a proper $PKUS$-algebra.

In the next part, we investigate some properties of $PKUS$-algebras.

Proposition 3.5. *Let X be a $PKUS$-algebra. Then the following propositions hold for any $x, y, z \in X$:*

(p_1) $x * (z \diamond x) = z \diamond 0$,
$x \diamond (z * x) = z * 0$.
(p_2) $(x * z) \diamond 0 = z * x$,
$(x \diamond z) * 0 = z \diamond x$.
(p_3) $0 * (0 \diamond x) = x$,
$0 \diamond (0 * x) = x$.
(p_4) $0 * (x \diamond z) = (0 * x) \diamond (0 * z)$,
$0 \diamond (x * z) = (0 \diamond x) * (0 \diamond z)$.
(p_5) $x * z = y * z \Rightarrow x * 0 = y * 0$,
$x \diamond z = y \diamond z \Rightarrow x \diamond 0 = y \diamond 0$.
(p_6) $x * [(x * z) \diamond z] = 0$,
$x \diamond [(x \diamond z) * z] = 0$.

Proof. Proofs of analogous properties are similar and so are omitted.

Let $x, y, z \in X$. Then
(p_1):By $(PKUS4)$, $x * (z \diamond x) = z \diamond 0$ as $x * x = 0$ from $(PKUS3)$.
(p_2): $(x * z) \diamond 0 = (x * z) \diamond (x * x)$ by $(PKUS3)$. Using $(PKUS1)$, we get, $(x * z) \diamond 0 = z * x$.
(p_3): Applying $(PKUS2)$ twice for $x \in X$ shows that $0 * (0 \diamond x) = x$.
(p_4): Put $y = 0$ in $(PKUS4)$. Then using $(PKUS2)$ to substitute x we get, $x \diamond (0 * z) = (0 * x) \diamond (0 * z)$. Hence, the proposition is proved.
(p_5): If $x * z = y * z$. Then $z \diamond (x * z) = z \diamond (y * z)$. It follows from $(PKUS4)$ that $x * (z \diamond z) = y * (z \diamond z)$. Hence, by $(PKUS3)$, we get $x * 0 = y * 0$.
(p_6): $x * [(x * z) \diamond z] = (x * z) \diamond (x * z) = 0$ by $(PKUS4)$ and $(PKUS3)$.

Corollary 3.6. *In a $PKUS$-algebra X, let $x \neq 0$. If $x*z = x$, then $x \diamond z \neq z$, for any $z \in X$.*

Proof. Suppose, to get a contradiction, that $x \diamond z = z$. Then $x * (x \diamond z) = x*z = x$. On the other hand, from $(PKUS4)$, $x*(x \diamond z) = x \diamond (x*z) = x \diamond x = 0$ which is a contradiction. Thus $x \diamond z \neq z$.

The next corollary is proved similarly.

Corollary 3.7. *In a $PKUS$-algebra X, let $x \neq 0$. If $x*z = z$, then $x \diamond z \neq x$, for any $z \in X$.*

In a $PKUS$-algebra X, we define a natural order $\leq$ as follow:

$$x \leq z \text{ if and only if } z \diamond x = 0 \text{ if and only if } z * x = 0.$$

This order relation is reflexive, antisymmetric and transitive.

Proposition 3.8. *In a $PKUS$-algebra X, define the following relation on X:*

$$x \leq z \iff z \diamond x = 0 \iff z * x = 0.$$

Then $(X, \leq, *, \diamond, 0)$ is a partially ordered set.

Proof. Let $x, y, z \in X$.
(i) We have $y \leq y$ since $y * y = y \diamond y = 0$.
(ii) If $x \leq z$ and $z \leq x$, then $z \diamond x = 0 = x \diamond z$. Then $x \diamond (z * x) = x \diamond (x * z)$. Using $(PKUS4)$, $z * (x \diamond x) = x * (x \diamond z)$, and so $z * 0 = x * 0$. Therefore, $(z * 0) \diamond 0 = (x * 0) \diamond 0$ and so $0 * z = 0 * x$ by (p_2). Thus, from $(PKUS2)$,

$z = x$.
(iii) If $x \leq z$ and $z \leq y$, then $z * x = 0$ and $y * z = 0$. Using $(PKUS3)$, $(PKUS1)$ and (p_3), we have $0 = (z*x) \diamond (z*x) = (z*x) \diamond [(y*z) \diamond (y*x)] = 0 \diamond [0 \diamond (y*x)] = y * x$ as $z * x = 0$ and $y * z = 0$. Therefore, $x \leq y$.

Proposition 3.9. *Let X be a PKUS-algebra. The following hold for any $x, y, z \in X$:*

(b_1) $z \leq x$ *and* $x \leq z \Rightarrow x = z$.
(b_2) $z \leq x \Rightarrow x * 0 = z * 0$,
$z \leq x \Rightarrow x \diamond 0 = z \diamond 0$,
$z \leq x \Rightarrow x * 0 = z \diamond 0$,
$z \leq x \Rightarrow x \diamond 0 = z * 0$.
(b_3) $x \leq 0 \Rightarrow x = 0$.
(b_4) $x \leq z \Rightarrow y * x \leq y * z$,
$x \leq z \Rightarrow y \diamond x \leq y \diamond z$.
(b_5) $x \leq z \Rightarrow z * y \leq x * y$,
$x \leq z \Rightarrow z \diamond y \leq x \diamond y$.
(b_6) $x * z \leq y \Rightarrow y \diamond z \leq x$,
$x \diamond z \leq y \Rightarrow y * z \leq x$.

Proof. Proofs of analogous properties are omitted. Let $x, y, z \in X$. Then
(b_1): Obvious from Proposition 3.8.
(b_2): If $z \leq x$ then $x * z = 0$. Thus $z * (x * z) = z * 0$. By using $(PKUS4)$, $z * (x * z) = x * 0$. Hence, $x * 0 = z * 0$.
(b_3): Direct from $(PKUS2)$.
(b_4): Let $x \leq z$. Thus, $z * x = 0$. On the other hand, from $(PKUS1)$, we know that $z*x = (y*z) \diamond (y*x)$. Hence, $(y*z) \diamond (y*x) = 0$ and so $y*x \leq y*z$.
(b_5): Let $x \leq z$. Thus $(PKUS1)$ becomes $(y * z) \diamond (y * x) = 0$. Multiplying both sides by 0 and using (p_2), we have $(y * x) \diamond (y * z) = 0$ and so $(y * z) \leq (y * x)$. Applying the previous step strategy again, the property is proved.
(b_6): The proof is direct using $(PKUS4)$.

In the next theorem we characterize $PKUS$-algebras.

Theorem 3.10. *A structure $(X, \leq, *, \diamond, 0)$, where X is a nonempty set, the relation $(\leq)$ is binary on X, the operations $*, \diamond$ are binary on X and 0 is an element of X, is a PKUS-algebra if the following hold:*

(1) $(z * y) \diamond (z * x) = y * x$,

$(z \diamond y) * (z \diamond x) = y \diamond x,$
(2) $y \leq x$ *and* $x \leq y \Rightarrow x = y,$
(3) $0 * x = x = 0 \diamond x,$
(4) $(x * y) \diamond [(y * z) \diamond (x * z)] = 0,$
$(x \diamond y) * [(y \diamond z) * (x \diamond z)] = 0.$

Proof. If axioms (1), (2), (3), (4) are satisfied, then it is clear that $(PKUS1)$ and $(PKUS2)$ are applicable. We can get $(PKUS3)$ using (4) and (3) as follows: put $x = 0, y = 0, z = x$ in (4). Then apply (3) to get $(0*0)\diamond[(0*x)\diamond(0*x)] = x \diamond x$. Hence $x \diamond x = 0$ and similarly $x * x = 0$. To get $(PKUS4)$, start with (4): $(x * y) \diamond [(y * z) \diamond (x * z)] = 0$ with $x = 0, y = z, z = x$. Then $(0*z)\diamond[(z*x)\diamond(0*x)] = 0$ implies $z\diamond[(z*x)\diamond x] = 0$ which implies $(z*x)\diamond x \leq z$. Multiply the inequality by $y*x$ and use (b_5) to get $z\diamond(y*x) \leq [(z*x)\diamond x]\diamond(y*x)$. Using (p_6) we have $z \diamond (y * x) \leq [(z * x) \diamond x] \diamond (y * x) \leq z \diamond (y * x)$. Applying $(PKUS4)$ to the right side, we get $z \diamond (y * x) \leq y * (z \diamond x)$. As both operations $\diamond, *$ are binary and z, y, x are arbitrary, the following inequality is true: $y*(z\diamond x) \leq z\diamond(y*x)$. Thus, $y*(z\diamond x) = z\diamond(y*x)$. Hence, $x*(y\diamond z) = y\diamond(x*z)$ and similarly $x \diamond (y * z) = y * (x \diamond z)$.

Definition 3.11. *Let X be a $PKUS$-algebra and let S be a nonempty subset of X. If $y*x \in S$ and $y\diamond x \in S$, for all $x, y \in S$, then S is a $PKUS$-subalgebra.*

Clearly, a $PKUS$-subalgebra is a $PKUS$-algebra.

Theorem 3.12. *For any $PKUS$-algebra X, the set of non negative elements of X is a $PKUS$-subalgebra of X denoted by $S(X)$.*

Proof. For $x, z \in S(X)$ we have $x \geq 0$ and $z \geq 0$. Using (b_4), $z*x \geq z*0$. As $z \geq 0$, we have $z * 0 = 0$ and so $z * x \geq 0$. Therefore, $z * x \in S(X)$. Similarly $z \diamond x \in S(X)$. Hence $S(X)$ is a $PKUS$-subalgebra of X.

Proposition 3.13. *In a $PKUS$-algebra X, if $x \in S(X)$ and $y \in X - S(X)$ (the complement of $S(X)$), then $x * y \in X - S(X)$ and $x \diamond y \in X - S(X)$.*

Proof. Suppose, to get a contradiction, that $x*y \in S(X)$. Then $x*y \geq 0$ which implies $y \geq x$. On the other hand, since $S(X)$ is a $PKUS$-subalgebra, $(x * y) \diamond 0 \in S(X)$. By (p_2), $y * x \in S(X)$ and so $y * x \geq 0$. Multiplying by x and using (b_5) we get $(y * x) \diamond x \leq 0 \diamond x$. Thus, from (p_6) and $(PKUS2)$, $y \leq x$. Hence $y = x$ and so $y \in S(X)$ which is a contradiction. Similarly, $x \diamond y \in X - S(X)$.

Next, the structure of pseudo KU-algebras is introduced and conditions are given to show when to get $PKUS$-algebras.

Definition 3.14. *For a non empty set X, a pseudo KU-algebra is a structure $(X, *, \diamond, 0)$ where $*, \diamond$ are binary operations on X and 0 is an element of X, the following hold:*

$(pKU1)$ $(x * z) \diamond [(z * y) \diamond (x * y)] = 0,$
$(x \diamond z) * [(z \diamond y) * (x \diamond y)] = 0,$
$(pKU2)$ $0 * x = x$ *and* $0 \diamond x = x,$
$(pKU3)$ $x * 0 = 0$ *and* $x \diamond 0 = 0,$
$(pKU4)$ $x * z = 0$ *and* $z * x = 0 \Rightarrow x = z,$
$x \diamond z = 0$ *and* $z \diamond x = 0 \Rightarrow x = z.$

Theorem 3.15. *If X is a pseudo KU-algebra satisfies the following conditions, for all $x, y, z \in X$:*

$$(z * y) \diamond (z * x) = y * x,$$

$$(z \diamond y) * (z \diamond x) = y \diamond x,$$

then X is a $PKUS$-algebra.

Proof. From [1, Lemma 2.2] and Theorem 3.10 the theorem follows.

4 The center of pseudo KUS-algebra

In this section, we give characterizations of minimal elements in $PKUS$-algebras. Then we introduce the center of $PKUS$-algebras.

Definition 4.1. *An element a in a $PKUS$-algebra X is said to be minimal if $x \leq a$ where $x \in X$ implies $x = a$.*

Proposition 4.2. *In a $PKUS$-algebra X, if a is a minimal element , then $(a \diamond x) * x = a$ and $(a * x) \diamond x = a$.*

Proof. The proof is straightforward from (p_6) and Definition 4.1.

Theorem 4.3. *In a $PKUS$-algebra X, the following statements are equivalent for $a, x \in X$:*

(1) $x \leq a$ *implies* $x = a,$
(2) $(a * x) \diamond x = a,$
(3) $(a * 0) \diamond 0 = a,$
(4) $0 \diamond (0 * a) = a,$
(5) $\exists\, x \in X : a = 0 \diamond x.$

Proof. (1) $\Rightarrow$ (2). Obvious from Proposition 4.2.
(2) $\Rightarrow$ (3). Obvious with $x = 0$.
(3) $\Rightarrow$ (4). Suppose $a = (a * 0) \diamond 0$. Using (p_2) and (p_3), $(a * 0) \diamond 0 = 0 * a = 0 \diamond (0 * a)$. Hence $0 \diamond (0 * a) = a$.
(4) $\Rightarrow$ (5). Suppose that $a = 0 \diamond (0 * a)$. By letting $a := 0 \diamond x$, and using (p_3) we have $a = 0 \diamond [0 * (0 \diamond x)] = 0 \diamond x$.
(5) $\Rightarrow$ (1). Suppose that for some $x \in X$, $a := 0 \diamond x$. To show that a is minimal, we start with $y \in X$ such that $y \leq a$ and show that $a \leq y$ as well. If $y \leq a$, then $a * y = 0$ which implies $(0 \diamond x) * y = 0$. Now We have $y * a = 0 \diamond (y * x)$ using $(PKUS4)$. From (p_4), $0 \diamond (y * x) = (0 \diamond x) * (0 \diamond y) = (0 \diamond x) * y$ using $(PKUS2)$. Thus $(0 \diamond x) * y = 0$ and so $y * a = 0$; i.e., $a \leq y$. Therefore a is minimal as desired (by Definition 4.1).

The analogous theorem to Theorem 4.3, where the operation $*$ is replaced by the operation $\diamond$ and vice versa, is proved similarly and so is omitted.

Following the terminology of [9], we set $\bar{x} = 0 \diamond (0 * x) = 0 * (0 \diamond x)$ for $x \in X$ where X is a $PKUS$-algebra, then we introduce the center of X.

Proposition 4.4. *In a $PKUS$-algebra X and for any $x, y \in X$, we have:*
(1) $\overline{x * y} = \bar{x} * \bar{y}$ *and* $\overline{x \diamond y} = \bar{x} \diamond \bar{y}$,
(2) $\bar{\bar{x}} = \bar{x}$.

Proof. (1): We have $\overline{x * y} = 0 * [0 \diamond (x * y)]$. Using (p_4) twice, $0 * [0 \diamond (x * y)] = 0 * [(0 \diamond x) * (0 \diamond y)] = [0 * (0 \diamond x)] * [0 * (0 \diamond y)] = \bar{x} * \bar{y}$. We can show similarly that, $\overline{x \diamond y} = \bar{x} \diamond \bar{y}$.
(2): We know that $\bar{\bar{x}} = 0 \diamond (0 * \bar{x})$ and $\bar{x} = 0 \diamond (0 * x)$. As $0 * x = 0 * [0 \diamond (0 * x)]$ by (p_3) $0 * x = 0 * \bar{x}$. Hence, $\bar{\bar{x}} = 0 \diamond (0 * x) = \bar{x}$.

Starting with $\overline{x * y} = 0 \diamond [0 * (x * y)]$ and $\overline{x \diamond y} = 0 \diamond [0 * (x \diamond y)]$ we get the next corollary.

Corollary 4.5. *In a $PKUS$-algebra X and for any $x, y \in X$, we have:*
$\overline{x * y} = \bar{x} \diamond \bar{y}$ *and* $\overline{x \diamond y} = \bar{x} * \bar{y}$.

In a $PKUS$-algebra X, we will call the set of all elements x in X where $x = \bar{x}$ the center of X and we will denote it by $Z(X)$. By Theorem 4.3, the center of X is the set of all minimal elements of X. Note that $0 \in Z(X)$ as $0 = \bar{0}$.

Proposition 4.6. *The center of a $PKUS$-algebra is a $PKUS$-subalgebra.*

Proof. Obvious from (1) in Proposition 4.4.

Proposition 4.7. *In a $PKUS$-algebra X, if $x, y \in Z(X)$, then for any $z \in X$, we have*

$$(x \diamond z) * y = (y * z) \diamond x.$$

Proof. We have $(x \diamond z) * y = (x \diamond z) * [(y * z) \diamond z]$ (using the assumption $y \in Z(X)$ and Theorem 4.3). Then using $(PKUS4)$, $(x \diamond z) * [(y * z) \diamond z] = (y * z) \diamond [(x \diamond z) * z] = (y * z) \diamond x$ (by the assumption $x \in Z(X)$ and Theorem 4.3).

For any element a in a $PKUS$-algebra X, define the set

$$V(a) = \{x \in X | a \leq x\}.$$

Note that $a \in V(a)$ as $a \leq a$, $V(0) = \{x \in X | 0 \leq x\} = S(X)$ and when $a \leq b$, $V(a) \subseteq V(b)$ for any $a, b \in X$.

Proposition 4.8. *Let X be a $PKUS$-algebra. For each $x \in X$, there exists a unique element $a \in Z(X)$, where $a \leq x$.*

Proof. Let $a = 0 \diamond (0 * x)$, where $x \in X$ and $a \in Z(X)$. Then $x \diamond a = x \diamond [0 \diamond (0 * x)] = x \diamond x = 0$ (using (p_3) and $(PKUS3)$). Hence $a \leq x$. To prove uniqueness, let $b \in Z(X)$ such that $b \leq x$. Then $x \diamond b = 0$ and $(x \diamond b) * b = 0 * b$. By Proposition 4.7, we have $(b * b) \diamond x = 0 * b$ and so $0 \diamond x = 0 * b$. Thus $0 * (0 \diamond x) = 0 * (0 * b) = b$. Hence, $a = b$ as needed.

Corollary 4.9. *In $PKUS$-algebras X, if $x, y \in Z(X)$ such that $x \neq y$, then $V(x) \bigcap V(y) = \phi$.*

Proof. Suppose, to get a contradiction, that $V(x) \bigcap V(y) \neq \phi$, where x, y are distinct elements in $Z(X)$. Then there exists an element $a \in V(x) \bigcap V(y)$ and so $x \leq a$ and $y \leq a$. By the uniqueness in Proposition 4.8 we get a contradiction. Thus, $V(x) \bigcap V(y) = \phi$.

5 The G-part of pseudo KUS-algebra

In this section, we give propositions of the G-part of $PKUS$-algebras.

Definition 5.1. *The G-part of a $PKUS$-algebras X is the set $\{x \in X | x * 0 = x$ and $x \diamond 0 = x\}$ denoted by $G(X)$.*

Proposition 5.2. *If X is a $PKUS$-algebra, then $y \in G(X)$ if and only if $x \diamond (y * x) = y$ and $x * (y \diamond x) = y$, for $x, y \in X$.*

Proof. For $y \in G(X)$, $y * 0 = y$. Using $(PKUS3)$, we write $y * 0 = y$ as $y * (x \diamond x) = y$. Thus $x \diamond (y * x) = y$ by $(PKUS4)$. Similarly, $x * (y \diamond x) = y$.

In $G(X)$, where X is a $PKUS$-algebra, right and left cancellation laws hold as we show in the following propositions.

Proposition 5.3. *Let X be $PKUS$-algebras. Then, for $x, y, z \in G(X)$, if $x * y = z * y$ and $x \diamond y = z \diamond y$, then $x = z$.*

Proof. Let $x * y = z * y$ and $x \diamond y = z \diamond y$, where $x, y, z \in G(X)$. Then by $(PKUS4)$ and $(PKUS3)$, $y \diamond (x * y) = x * (y \diamond y) = x * 0$ and $y \diamond (z * y) = z * (y \diamond y) = z * 0$. Since $x * y = z * y$ then $x * 0 = z * 0$ which implies $x = z$ as $x, z \in G(X)$.

Corollary 5.4. *Let X be a $PKUS$-algebra. Then $x * z = z \diamond x$, for any $x, z \in G(X)$.*

Proof. For $x, z \in G(X)$, $x * z = x * (z \diamond 0) = z \diamond (x * 0) = z \diamond x$.

Proposition 5.5. *In a $PKUS$-algebra X, if $y * x = y * z$ and $y \diamond x = y \diamond z$, then $x = z$, for $x, y, z \in G(X)$.*

Proof. Let $x, y, z \in G(X)$ with $y * x = y * z$ and $y \diamond x = y \diamond z$. As $x \in G(X)$, $x = x \diamond 0 = y * (x \diamond y)$ using (p_1). Then, by Corollary 5.4, $y * (x \diamond y) = y * (y * x) = y * (y * z) = y * (z \diamond y) = z \diamond (y * y) = z \diamond 0 = z$.

Proposition 5.6. *For $x, y, z \in G(X)$, where X is a $PKUS$-algebra, if $x \diamond z = y$, then $z * y = x$.*

Proof. Given that $y = x \diamond z$ and using $(PKUS4)$ and $(PKUS3)$, we have, $z * y = z * (x \diamond z) = x \diamond (z * z) = x \diamond 0 = x$, as $x \in G(X)$.

Proposition 5.7. *Let X be $PKUS$-algebras. Then for $x, z \in G(X)$, $x * (0 \diamond z) = z * (0 \diamond x)$.*

Proof. For $x, z \in G(X)$, we have by Corollary 5.4, $x * (0 \diamond z) = x * (z * 0) = z * (x * 0)$ by $(PKUS4)$. Then, by Corollary 5.4, $z * (x * 0) = z * (0 \diamond x)$.

Definition 5.8. *Let X be a $PKUS$-algebra. The p-radical of X is the set $\{x \in X | x * 0 = 0$ and $x \diamond 0 = 0\}$ denoted by $P(X)$.*

Proposition 5.9. *If X is a $PKUS$-algebra. Then $y \in P(X)$ if and only if, for any $x \in X$, $x \diamond (y * x) = 0$ and $x * (y \diamond x) = 0$.*

Proof. By $(PKUS4)$ and $(PKUS3)$, we have $x \diamond (y * x) = y * (x \diamond x) = y * 0 = 0 \iff y \in P(X)$. The proof of the analogous part is similar and so is omitted.

References

[1] S. M. Mostafa, M. A. Abdel Naby, F. A. Halim, A. T. Hameed, On KUS-Algebras, International Journal of Algebra, **7,** no. 3, (2013), 131-144.

[2] C. Prabpayak, U. Leerawat, On ideals and congruences in KU-algebras, Scientia Magna, **5,** no. 1, (2009), 54-57.

[3] Y. Imai, K. Iséki, On axiom systems of propositional calculi. XIV, Proceedings of the Japan Academy, **42,** no. 1, (1966), 19-22.

[4] K. Iséki, An algebra related with a propositional calculus, Proceedings of the Japan Academy, **42,** no. 1, (1966), 26-29.

[5] G. Georgescu, A. Iorgulescu, Pseudo-BCK algebras: an extension of BCK algebras, Combinatorics, computability and logic. Springer London, (2001), 97-114.

[6] W. A. Dudek, Y. B. Jun, Pseudo-BCI algebras, East Asian mathematical journal, **24,** no. 2, (2008), 187-190.

[7] Y. H. Kim, K. S. So, On minimality in pseudo-BCI-algebras, Communications of the Korean Mathematical Society, **27,** no. 1, (2012), 7-13.

[8] G. Dymek, Atoms and ideals of pseudo-BCI-algebras, Commentationes Mathematicae, **52,**, (2012), 73-90.

[9] A. Walendziak, Pseudo-BCH-algebras, Discussiones Mathematicae-General Algebra and Applications, **35,** no. 1, (2015), 5-19.

[10] Y. B. Jun, S. S. Ahn, On Pseudo BCH-algebras, Applied Mathematical Sciences, **9,** no. 39, (2015), 1931-1939.

[11] R. A. Borzooei, A. B. Saeid, A. Rezaei, A. Radfar, R. Ameri, On pseudo BE-algebras, Discussiones Mathematicae-General Algebra and Applications, **33,** no. 1, (2013), 95-108.

[12] G. Georgescu, A. Iorgulescu, Pseudo-BL algebras: A noncommutative extension of BL algebras, In fifth international conference FSTA, (2000), 90-92.

[13] Y. B. Jun, H. S. Kim, J. Neggers, Pseudo d-algebras, Information Sciences, **179,** no. 11, (2009), 1751-1759.