International Journal of Mathematics and
Computer Science, **16**(2021), no. 1, 321–332

A Subclass of Odd p-valent Functions by Salagean Operator with Negative Coefficients

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(Received July 4, 2020, Accepted July 29, 2020)

Abstract

In this paper, much emphasis is on the class of analytic function which is p-valent in the open unit disc. The main objective is to study the new subclasses of p-valent odd function defined using Salagean differential operator. In this work, we determine the estimates for the coefficients bound of the functions in this class. Relevant connections with earlier known result are made.

1 Introduction

Let $A(p)$ denote the class of normalized analytic functions f of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, (p \in \mathbb{N} = \{1, 2, 3, ...\}) \tag{1.1}$$

Key words and phrases: Odd p-valent function, Negative Coefficients, Salagean Derivative Operator.
AMS (MOS) Subject Classifications: 30C45.
ISSN 1814-0432, 2021, http://ijmcs.future-in-tech.net

where $z \in \mathcal{U} = \{z : |z| < 1\}$, and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. We denote $T(p)$ be the subclass of $A(p)$ consisting of analytic and p-valent in $\mathcal{U}$ with negative coefficients functions f of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} |a_k| z^k, (p \in \mathbb{N} = \{1, 2, 3, ...\}). \tag{1.2}$$

For each function $f \in A(1)$, analytic and univalent in $\mathcal{U}$. The square-root transform is an odd univalent function given by

$$h(z) = \sqrt{h(z)} = z + c_3 z^3 + c_5 z^5 + \cdots = z + \sum_{k=2}^{\infty} c_k z^k. \tag{1.3}$$

The set of all odd functions in $A(1)$ is denoted by $A(1)^{(2)}$.

More generally, for each of integer $m \geq 2$, the class of all mth-root transforms

$$h(z) = \{f(z^m)\}^{\frac{1}{m}} = z + c_{m+1} z^{m+1} + c_{2m+1} z^{2m+1} + \cdots \tag{1.4}$$

of function $f \in A(1)$ is denoted by $A(1)^{(m)}$. This is precisely the set of all functions $h \in A(1)$ with m-fold symmetry. It is easy to see that the square-root transform of Koebe function is

$$\frac{z}{1 - z^2} = z + z^3 + z^5 + \cdots . \tag{1.5}$$

It is to be expected that this function will play the role of the Koebe function in the class $A(1)^{(2)}$. See Duren [1]

Then, we let $A_c(p)$ analytic and p-valent odd function in the class of the form:

$$h(z) = z^p + \sum_{k=p+1}^{\infty} c_{2k-1} z^{2k-1}, (p \in \mathbb{N} = \{1, 2, 3, ...\}) \tag{1.6}$$

analytic in $\mathcal{U}$. Let $T_c(p)$ for $A_c(p)$ analytic and p-valent odd function of negative coefficients in $\mathcal{U}$ of the form:

$$h(z) = z^p - \sum_{k=p+1}^{\infty} |c_{2k-1}| z^{2k-1}. \tag{1.7}$$

Now, we denote by $S^*(p, \alpha), K(p, \alpha), 0 \leq \alpha \leq p$, the class of p-valently starlike functions of order α and the class of p-valently convex of order α respectively, where

$$S^*(p, \alpha) = \left\{ h \in A_c(p) : \ \mathrm{Re} \left\{ \frac{zh'(z)}{h(z)} \right\} > \alpha, \quad z \in \mathcal{U} \right\} \tag{1.8}$$

$$K(p,\alpha)=\left\{h\in A_c(p):\ \mathrm{Re}\left\{1+\frac{zh''(z)}{h'(z)}\right\}>\alpha,\quad z\in\mathcal{U}\right\} \tag{1.9}$$

We also denote by $T^*(p,\alpha)$ and $C(p,\alpha)$ the subclass of $T_c(p)$ that are respectively, p-valently starlike of order α and p-valently convex of order α with negative coefficients.

Note that the functions $f\in T(1)$ that are starlike of order α, and convex of order $\alpha,(0\le\alpha\le 1)$ have been studied by Silverman [3].

Now, let functions $h\in A_c(p)$. By using the approach introduced by Salagean [2], we introduced the following operator:

$$\begin{aligned}
D^0h(z) &= h(z)=z+c_3z^3+c_5z^5+\cdots=z+\sum_{k=2}^{\infty}c_{2k-1}z^{2k-1}\\
D^1h(z) &= Dh(z)=zh'(z)=z+3c_3z^3+5c_5z^5+\cdots=z+\sum_{k=2}^{\infty}(2k-1)c_{2k-1}z^{2k-1}\\
&\cdots\\
D^nh(z) &= D(D^{n-1}h(z))=z+\sum_{k=2}^{\infty}(2k-1)^nc_{2k-1}z^{2k-1}
\end{aligned}$$

Thus, we write p-valent odd function defined by Salagean derivative operator as

$$\mathcal{H}(n,z)=z^p+\sum_{k=p+1}^{\infty}(2k-1)^nc_{2k-1}z^{2k-1} \tag{1.10}$$

Definition 1.1. *The function $h\in A_c(p)$, is said to be in the class $S_n(p,\alpha)$ if it is satisfies*

$$\mathrm{Re}\left\{\frac{D^{n+1}h'(z)}{D^nh(z)}-p\right\}>\alpha\quad for\quad (0\le\alpha\le p)\quad and\quad n\in\mathbb{N}) \tag{1.11}$$

We note that, $S_0(p,\alpha)=S^*(p,\alpha)$ and $S_1(p,\alpha)=K(p,\alpha)$, since

$$\frac{Dh'(z)}{h(z)}=\frac{zh'(z)}{h(z)}\quad and\quad \frac{D^2h'(z)}{Dh(z)}=\frac{z(zh'(z))'}{zh'(z)}, \tag{1.12}$$

respectively.

Definition 1.2. *Let $h\in A_c(p)$, then*

$$T_n(p,\alpha)=\left\{h\in T_c(p):\ \mathrm{Re}\left\{\frac{D^{n+1}h'(z)}{D^nh(z)}-p\right\}>\alpha,\quad z\in\mathbb{U}\right\}. \tag{1.13}$$

We note that, $T_0(p,\alpha) = T^*(p,\alpha)$ and $T_1(p,\alpha) = C(p,\alpha)$. Therefore, $T_n(p,\alpha) \subset S_n(p,\alpha)$. Following the earlier works by Siregar & Darus [4], we begin by finding the coefficient inequalities, distortion and covering theorem and linear convex combination of the studied function.

2 Coefficient Inequalities

Theorem 2.1. *Let $h(z)$ given by (1.6), and*

$$\sum_{k=p+1}^{\infty} [(2k-1)^{n+1} - (2k-1)^n \alpha]|c_{2k-1}| \leq p^n(p-\alpha)$$

then $h \in S_n(p,\alpha)$.

Proof. It is suffices to show that

$$\left| \frac{D^{n+1}h(z)}{D^n h(z)} - p \right| \leq p - \alpha.$$

we have

$$\begin{aligned} & \left| \frac{D^{n+1}h(z)}{D^n h(z)} - p \right| \\ = & \left| \frac{D^{n+1}h(z) - pD^n h(z)}{D^n h(z)} \right| \\ = & \left| \frac{p^{n+1}z^p + \sum_{k=p+1}^{\infty}(2k-1)^{n+1}c_{2k-1}z^{2k-1} - p\left(p^n z^p + \sum_{k=p+1}^{\infty}(2k-1)^n c_{2k-1}z^{2k-1}\right)}{p^n z^n + \sum_{k=p+1}^{\infty}(2k-1)^n c_{2k-1}z^{2k-1}} \right| \\ \leq & \frac{\sum_{k=p+1}^{\infty}[(2k-1)^{n+1} - p(2k-1)^n)|c_{2k-1}||z^{2k-1}|}{p^n|z|^p + \sum_{k=p+1}^{\infty}(2k-1)^n|c_{2k-1}||z^{2k-1}|}, \quad |z| = 1 \\ \leq & \frac{\sum_{k=p+1}^{\infty}[(2k-1)^{n+1} - p(2k-1)^n)|c_{2k-1}|}{p^n + \sum_{k=p+1}^{\infty}(2k-1)^n|c_{2k-1}|} \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{k=p+1}^{\infty} [(2k-1)^{n+1} - \alpha(2k-1)^n)|c_{2k-1}| &\leq p^n(p-\alpha) \\ \Rightarrow \sum_{k=p+1}^{\infty} [(2k-1)^{n+1} - p(2k-1)^n + p(2k-1)^n - \alpha(2k-1)^n]|c_{2k-1}| &\leq p^n(p-\alpha) \end{aligned}$$

$$\Rightarrow \sum_{k=p+1}^{\infty} [(2k-1)^{n+1} - p(2k-1)^n]|c_{2k-1}| \leq p^n(p-\alpha) - \sum_{k=p+1}^{\infty} [p(2k-1)^n - \alpha(2k-1)^n]\,|c_{2k-1}|$$

$$\Rightarrow \sum_{k=p+1}^{\infty} [(2k-1)^{n+1} - p(2k-1)^n]|c_{2k-1}| \leq (p-\alpha)\left[p^n - \sum_{k=p+1}^{\infty} p(2k-1)^n|c_{2k-1}|\right]$$

$$\Rightarrow \frac{\sum_{k=p+1}^{\infty}[(2k-1)^{n+1} - p(2k-1)^n)|c_{2k-1}|}{p^n - \sum_{k=p+1}^{\infty} p(2k-1)^n|c_{2k-1}|} \leq p^n(p-\alpha).$$

Thus we obtain

$$\left|\frac{D^{n+1}h(z)}{D^n h(z)} - p\right| \leq p^n(p-\alpha),$$

and the proof is complete.

□

Letting $n = 0$ and $n = 1$ in Theorem 2.1 respectively, we have the following corollaries

Corollary 2.2. *Let $h(z)$ given by (1.6), then*

i. $h \in S^*(p,\alpha)$ *if and only if* $\sum_{k=2}^{\infty}(2k-1-\alpha)|c_{2k-1}| \leq p-\alpha$

ii. $h \in K(p,\alpha)$ *if and only if* $\sum_{k=2}^{\infty}[(2k-1)^2 - \alpha(2k-1)]|c_{2k-1}| \leq p(p-\alpha)$.

Letting $p = 1$ in Corollary 2.2 yields

Corollary 2.3. *Let $h(z)$ given by (1.6), then*

i. $h \in S^*(\alpha)$ *if and only if* $\sum_{k=2}^{\infty}(2k-1-\alpha)|c_{2k-1}| \leq 1-\alpha$

ii. $h \in K(\alpha)$ *if and only if* $\sum_{k=2}^{\infty}[(2k-1)^2 - \alpha(2k-1)]|c_{2k-1}| \leq 1-\alpha$.

Theorem 2.4. *Let $h(z)$ given by (1.7), then $h \in T_n(p,\alpha)$ if and only if*

$$\sum_{k=p+1}^{\infty} [(2k-1)^{n+1} - (2k-1)^n\alpha]|c_{2k-1}| \leq p^n(p-\alpha).$$

Proof. In view of Theorem 2.1, it is suffices to show the only if part. Assume that

$$\text{Re}\left\{\frac{D^{n+1}h(z)}{D^n h(z)}\right\} = \text{Re}\left\{\frac{p^{n+1}z^p - \sum_{k=p+1}^{\infty}(2k-1)^{n+1}|c_{2k-1}|z^{2k-1}}{p^n z^p - \sum_{k=p+1}^{\infty}(2k-1)^n|c_{2k-1}|z^{2k-1}}\right\} > \alpha.$$

We choose value of on the real axis so that $\frac{D^{n+1}h(z)}{D^n h(z)}$ is real.
Letting $z \to 1$ through real values, we have

$$\begin{aligned} p^{n+1} - \sum_{k=p+1}^{\infty}(2k-1)^{n+1}|c_{2k-1}| &\geq \alpha\left\{p^n + \sum_{k=p+1}^{\infty}(2k-1)^n|c_{2k-1}|\right\} \\ \Rightarrow -\sum_{k=p+1}^{\infty}(2k-1)^{n+1}|c_{2k-1}| + \sum_{k=p+1}^{\infty}\alpha(2k-1)^n|c_{2k-1}| &\geq -p^n(p-\alpha) - \alpha p^n \\ \Rightarrow \sum_{k=p+1}^{\infty}[(2k-1)^{n+1} - \alpha(2k-1)^n]|c_{2k-1}| &\leq p^n(p-\alpha). \end{aligned}$$

Thus we obtain

$$\sum_{k=p+1}^{\infty}[(2k-1)^{n+1} - \alpha(2k-1)^n]|c_{2k-1}| \leq p^n(p-\alpha).$$

and the proof is complete.

□

Letting $n = 0$ and $n = 1$ in Theorem 2.4, we have the following corollaries

Corollary 2.5. *Let $h(z)$ given by (1.7), then*

i. $h \in T^*(p,\alpha)$ *if and only if* $\sum_{k=2}^{\infty}(2k-1-\alpha)|c_{2k-1}| \leq p-\alpha$

ii. $h \in C(p,\alpha)$ *if and only if* $\sum_{k=2}^{\infty}[(2k-1)^2 - \alpha(2k-1)]|c_{2k-1}| \leq p(p-\alpha)$

Letting $p = 1$ in Corollary 2.5 yields

Corollary 2.6. *Let $h(z)$ given by (1.7), then*

i. $h \in S^*(\alpha)$ *if and only if* $\sum_{k=2}^{\infty}(2k-1-\alpha)|c_{2k-1}| \leq 1-\alpha$

ii. $h \in K(\alpha)$ *if and only if* $\sum_{k=2}^{\infty}[(2k-1)^2 - \alpha(2k-1)]|c_{2k-1}| \leq 1-\alpha.$

3 Distortion and Covering Theorem for $T_n(p,\alpha)$

Theorem 3.1. *If the function $h(z)$ given by (1.7) is in the class $T_n(p,\alpha)$ for $0 < |z| = r < 1$, then for $|z| = r$,*

$$r^p - \frac{p^n(p-\alpha)}{((2p+1)^{n+1} - (2p+1)^n\alpha)}r^{2p+1} \leq |h(z)| \leq r^p + \frac{p^n(p-\alpha)}{((2p+1)^{n+1} - (2p+1)^n\alpha)}r^{2p+1}$$

with the equality holding at $z = \pm r$ for

$$h(z) = r^p - \frac{p^n(p-\alpha)}{((2p+1)^{n+1} - (2p+1)^n\alpha)} r^{2p+1}$$

Proof. We have

$$[(2p+1)^{n+1} - \alpha(2p+1)^n] \sum_{k=p+1}^{\infty} |c_{2k-1}| \leq \sum_{k=p+1}^{\infty} [(2k-1)^{n+1} - \alpha(2k-1)^n] |c_{2k-1}| \leq p^n(p-\alpha).$$

From Theorem 2.1,

$$|c_{2k-1}| \leq \frac{p^n(p-\alpha)}{[(2k-1)^{n+1} - \alpha(2k-1)^n]}$$

and for $k = p+1$, we obtain

$$|c_{2p+1}| \leq \frac{p^n(p-\alpha)}{[(2p+1)^{n+1} - \alpha(2p+1)^n]}.$$

Thus

$$\begin{aligned} |h(z)| &= \left| z^p + \sum_{k=p+1}^{\infty} c_{2k-1} \right|, \quad (|z| = r), \\ &\leq r^p + \sum_{k=p+1}^{\infty} |c_{2k-1}| r^{2k-1} \\ &\leq r^p + r^{2p+1} |a_{2p+1}| \\ &\leq r^p + \frac{p^n(p-\alpha)}{[(2p+1)^{n+1} - \alpha(2p+1)^n]}. \end{aligned}$$

and

$$\begin{aligned} |h(z)| &= z^p - \sum_{k=p+1}^{\infty} |c_{2k-1}|, \quad (|z| = r), \\ &\geq r^p - \sum_{k=p+1}^{\infty} |c_{2k-1}| r^{2k-1} \\ &\geq r^p - r^{2p+1} |a_{2p+1}| \\ &\geq r^p - \frac{p^n(p-\alpha)}{[(2p+1)^{n+1} - \alpha(2p+1)^n]}. \end{aligned}$$

Thus we obtain

$$r^p - \frac{p^n(p-\alpha)}{((2p+1)^{n+1}-(2p+1)^n\alpha)}r^{2p+1} \leq |h(z)| \leq r^p + \frac{p^n(p-\alpha)}{((2p+1)^{n+1}-(2p+1)^n\alpha)}r^{2p+1}$$

The proof is complete.

□

Letting $n = 0$ and $n = 1$ in Theorem 3.1 give the following corollaries

Corollary 3.2. *If the function h defined by (1.7), for $0 < |z| = r < 1$, then*

i. $h \in T^*(p, \alpha)$ *if and only if*

$$r^p - \frac{p-\alpha}{(2p+1-\alpha)}r^{2p+1} \leq |h(z)| \leq r^p + \frac{p-\alpha}{(2p+1-\alpha)}r^{2p+1}$$

ii. $h \in K(p, \alpha)$ *if and only if*

$$r^p - \frac{p(p-\alpha)}{[(2p+1)^2-(2p+1)\alpha]}r^{2p+1} \leq |h(z)| \leq r^p + \frac{p(p-\alpha)}{[(2p+1)^2-(2p+1)\alpha]}r^{2p+1}$$

Letting $p = 1$ in Corollary 3.2 yields

Corollary 3.3. *If the function h defined by (1.7), for $0 < |z| = r < 1$, then*

i. $h \in T^*(\alpha)$ *if and only if* $r - \frac{1-\alpha}{3-\alpha}r^3 \leq |h(z)| \leq r + \frac{1-\alpha}{3-\alpha}r^3$

ii. $h \in K(\alpha)$ *if and only if* $r - \frac{1-\alpha}{3(3-\alpha)}r^3 \leq |h(z)| \leq r + \frac{1-\alpha}{3(3-\alpha)}r^3$.

Theorem 3.4. *The disk $\{z : |z| < 1\}$ is mapped onto domain that contains the disk with*

$$|h(z)| < 1 - \frac{p^n(p-\alpha)}{[(2p+1)^{n+1}-(2p+1)^n\alpha]}$$

for any $h \in T_n(p, \alpha)$.

Proof. The proof follows upon letting $r \to 1$ in Theorem 3.4. □

When we let $n = 0$ and $n = 1$ in Theorem 3.4. We have the following corollaries

Corollary 3.5. *The disk $\{z : |z| < 1\}$ is mapped onto domain that contains the disk with*

i. $|h| \leq \frac{p}{2p+1-\alpha}$ *for any* $h \in T^*(p,\alpha)$, *and*

ii. $|h| \leq 1 - \frac{p(p-\alpha)}{(2p+1)(2p+1-\alpha)}$ *for any* $h \in C(p,\alpha)$.

Letting $p = 1$ in Corollary 3.8. We have

Corollary 3.6. *The disk* $\{z : |z| < 1\}$ *is mapped onto domain that contains the disk with*

i. $|h| \leq \frac{1}{3-\alpha}$ *for any* $h \in T^*(\alpha)$, *and*

ii. $|h| \leq 1 - \frac{1-\alpha}{3(3-\alpha)}$ *for any* $h \in C(\alpha)$.

Theorem 3.7. *If the function* $h(z)$ *given by (1.7) is in the class* $T_n(p,\alpha)$ *for* $0 < |z| = r < 1$, *then for* $|z| = r$,

$$pr^{p-1} - \frac{p^n(p-\alpha)}{[(2p+1)^n - (2p+1)^{n-1}\alpha]} r^{2p} \leq |h(z)| \leq pr^{p-1} - \frac{p^n(p-\alpha)}{[(2p+1)^n - (2p+1)^{n-1}\alpha]} r^{2p}.$$

Proof. We have

$$|h'(z)| \leq p|z|^{p-1} + \sum_{k=p+1}^{\infty} (2k-1)|c_{2k-1}||z|^{2k-2} \leq pr^{p-1} + (2p+1)r^{2p} \sum_{k=p+1}^{\infty} (2k-1)|c_{2k-1}|.$$

From Theorem 2.1,

$$\sum_{k=p+1}^{\infty} |c_{2k-1}| \leq \frac{p^n(p-\alpha)}{[(2k-1)^{n+1} - \alpha(2k-1)^n]}$$

Thus

$$|h'(z)| \leq pr^{p-1} + \frac{p^n(p-\alpha)}{[(2p+1)^n - \alpha(2p+1)^{n-1}]} r^{2p}.$$

and using the same argument, we have

$$|h'(z)| \geq pr^{p-1} - \frac{p^n(p-\alpha)}{[(2p+1)^n - \alpha(2p+1)^{n-1}]} r^{2p}.$$

Thus we obtain

$$pr^{p-1} - \frac{p^n(p-\alpha)}{[(2p+1)^n - \alpha(2p+1)^{n-1}]} r^{2p} \leq |h'(z)| \leq pr^{p-1} + \frac{p^n(p-\alpha)}{[(2p+1)^n - \alpha(2p+1)^{n-1}]} r^{2p}$$

This completes the proof.

□

Letting $n = 0$ and $n = 1$ in Theorem 3.1 give the following corollaries

Corollary 3.8. *If the function h defined by (1.7), for $0 < |z| = r < 1$, then*

i. $h \in T^*(p, \alpha)$ *if and only if*

$$pr^{p-1} + \frac{p-\alpha}{\alpha(p+1)^{-1}} r^{2p} \leq |h'(z)| \leq pr^{p-1} - \frac{p-\alpha}{\alpha(p+1)^{-1}} r^{2p}$$

ii. $h \in K(p, \alpha)$ *if and only if*

$$pr^{p-1} - \frac{p(p-\alpha)}{2p+1-\alpha} r^{2p} \leq |h'(z)| \leq pr^{p-1} + \frac{p(p-\alpha)}{2p+1-\alpha} r^{2p}$$

Letting $p = 1$ in Corollary 3.8 yields

Corollary 3.9. *If the function h defined by (1.7), for $0 < |z| = r < 1$, then If the function h defined by (1.7), for $0 < |z| = r < 1$, then*

i. $h \in T^*(\alpha)$ *if and only if*

$$r^{-1} + \frac{1-\alpha}{\alpha(2)^{-1}} r^{2p} \leq |h'(z)| \leq r^{-1} + \frac{1-\alpha}{\alpha(2)^{-1}} r^{2p}$$

ii. $h \in K(\alpha)$ *if and only if*

$$r^{-1} - \frac{1-\alpha}{3-\alpha} r^{2} \leq |h'(z)| \leq r^{-1} - \frac{1-\alpha}{3-\alpha} r^{2}.$$

4 Linear Convex Combinations

Theorem 4.1. *Let h defined by (1.7) be in the class $T_n(p, \alpha)$. Then $T_n(p, \alpha)$ is closed under linear convex combinations.*

Proof. Suppose that the functions h and g are defined by

$$h(z) = z^p - \sum_{k=p+1}^{\infty} c_{2k-1} z^{2k-1}, \quad (c_{2k-1} \geq 0)$$

and

$$g(z) = z^p - \sum_{k=p+1}^{\infty} b_{2k-1} z^{2k-1}, \quad (b_{2k-1} \geq 0).$$

respectively and assume that h and g are in the class $T_n(p,\alpha)$, we want to prove that the function $G(z)$ defined by

$$G(z) = (1-\lambda)h(z)+\lambda g(z) = z^p - \sum_{k=p+1}^{\infty} d_{2k-1}z^{2k-1}, \quad (d_{2k-1} \geq 0, \quad 0 \leq \lambda \leq 1).$$

is also in the class $T_n(p,\alpha)$. Since

$$\sum_{k=p+1}^{\infty} [(2k-1)^{n+1} - \alpha(2k-1)^n]c_{2k-1} \leq p^n(p-\alpha)$$

and

$$\sum_{k=p+1}^{\infty} [(2k-1)^{n+1} - \alpha(2k-1)^n]b_{2k-1} \leq p^n(p-\alpha)$$

with the aid of Theorem 2.1, we have

$$\begin{aligned}
\sum_{k=p+1}^{\infty} [(2k-1)^{n+1} - \alpha(2k-1)^n]c_{2k-1} &= \sum_{k=p+1}^{\infty} [(2k-1)^{n+1} - \alpha(2k-1)^n][(1-\lambda)c_{2k-1} + \lambda b_{2k-1}] \\
&= (1-\lambda)\sum_{k=p+1}^{\infty} [(2k-1)^{n+1} - \alpha(2k-1)^n]c_{2k-1} \\
&\quad +\lambda \sum_{k=p+1}^{\infty} [(2k-1)^{n+1} - \alpha(2k-1)^n]b_{2k-1} \\
&\leq (1-\lambda)p^n(p-\alpha) + \lambda p^n(p-\alpha) = p^n(p-\alpha).
\end{aligned}$$

Hence the theorem follows. □

Acknowledgement. The work here is under Universiti Kebangsaan Malaysia's research Grant: GUP-2019-032.

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