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On a Class of Analytic Functions Related to Close-to-Convex Functions

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Abstract

Making use of a q-differential operator, we introduce a new subclass of close-to-convex and derive various coefficient estimates for the class defined in the unit disc. Moreover, we investigate various properties like the inclusion relationship, distortion and radius of convexity. Furthermore, we apply our results highlighting the relevant connections with various other known results.

1 Introduction

Denote by $\mathcal{A}$ the class of functions having a Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathcal{U} = \{z : |z| < 1\}). \tag{1.1}$$

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Let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$ and let $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ denote the well known classes of starlike, convex and close-to-convex function respectively. For $0 \leq \alpha < 1$, $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ symbolize the classes of starlike functions of order α and convex functions of order α respectively. The class $\mathcal{P}$ denotes the class of functions of the form $p(z) = 1+p_1z+p_2z^2+p_3z^3+\cdots$ that are analytic in $\mathcal{U}$ and such that $Re\,(p(z)) > 0$ for all z in $\mathcal{U}$.

Let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$ if there exists an Schwartz function $w(z)$ in $\mathcal{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

In [3], Gao and Zhou introduced a class $\mathcal{K}_s$ which satisfies the analytic condition

$$Re\left(\frac{z^2 f'(z)}{g(z)g(-z)}\right) < 0 \quad \left(g \in \mathcal{S}^*\left(1/2\right);\, f \in \mathcal{A}\right), \tag{1.2}$$

and they proved that $\mathcal{K}_s$ is a subclass of close-to-convex function. The class $\mathcal{K}_s$ was further extended to a class $\mathcal{K}_s(\alpha)$ which was a natural generalization obtained by replacing "0" on the right hand side of (1.2) by α . The class $\mathcal{K}_s(\alpha)$ was studied by Kowalczyk and Leś-Bomba in [12].

Recently, Singh et al. [20] introduced the following class using subordination. A function $f \in \mathcal{A}$ is said to in $X_t\,(A,\, B)\,, |t| \leq 1,\, t \neq 0$ if there exists a function $g \in \mathcal{S}^*(1/2)$ such that

$$\frac{z^2 f'(z)}{g(z)g(tz)} \prec \frac{1+Az}{1+Bz}, \quad (-1 \leq B < A \leq 1,\, z \in \mathcal{U}).$$

The class $X_t\,(1-2\alpha,\, -1) = X_t(\alpha)$ was introduced and studied by Prajapat in [19].

Now we give a very brief introduction on q-calculus and the notations which are required for our study.

Quantum calculus, mostly known as q-calculus, is based on the idea of finite difference re-scaling. The difference of quantum differentials from the ordinary ones is that the notion of limit is removed in q-calculus; that is, the q-derivative is merely a ratio given by

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}.$$

Notice that as $q \to 1^-$, $\lim D_q f(z) = f'(z)$. q-calculus has numerous applications in a variety of disciplines such as special functions, operator

theory, quantum-mechanics, relativity, etc.
Throughout this paper, we let

$$[n]_q = \sum_{k=1}^{n} q^{k-1}, \quad [0]_q = 0, \quad (q \in \mathbb{C})$$

and the q-shifted factorial by

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & n = 1, 2, \dots . \end{cases}$$

The q-hypergeometric series was developed by Heine as a generalization of the hypergeometric series

$${}_2F_1[a, b;\, c|q, z] = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n. \tag{1.3}$$

Generalizing the Heine's series, we define ${}_r\phi_s$ the basic hypergeometric series by

$${}_r\phi_s\,(a_1,\, a_2,\, \dots,\, a_r;\, b_1,\, b_2,\, \dots,\, b_s; q,\, z) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \dots (a_r;q)_n}{(q;q)_n (b_1;q)_n \dots (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n, \tag{1.4}$$

with $\binom{n}{2} = \frac{n(n-1)}{2}$, where $q \neq 0$ when $r > s+1$. In (1.3) and (1.4), it is assumed that the parameters $b_1,\, b_2,\, \dots,\, b_s$ are such that the denominator factors in the terms of the series are never zero.

For complex parameters $a_1,\, \dots,\, a_r$ and $b_1, \dots,\, b_s$, $\left(\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\, \mathbb{Z}_0^- = 0, -1,\, -2,\, \dots;\right.$ $j = 1,\, \dots,\, s)$, we define the generalized q-hypergeometric function ${}_r\Psi_s(a_1,\, \dots,\, a_q;\, b_1,\, \dots,\, b_s;\, q, z)$ by

$${}_r\Psi_s(a_1,\, a_2,\, \dots,\, a_r;\, b_1,\, b_2,\, \dots,\, b_s;\, q, z) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \dots (a_r;q)_n}{(q;q)_n (b_1;q)_n \dots (b_s;q)_n} z^n, \tag{1.5}$$

$$(r = s+1;\, r,\, s \in \mathcal{N}_0 = \mathcal{N} \cup \{0\};\, z \in \mathcal{U}),$$

where $\mathcal{N}$ denotes the set of positive integers. By using the ratio test, if $|q| < 1$, the series (1.5) converges absolutely for $|z| < 1$ and $r = s+1$. For more mathematical background of these functions, the reader may refer to [2].

The function $\mathcal{G}_{r,s}(a_i, b_j; q, z)$ $(i = 1, 2, \ldots, r; j = 1, 2, \ldots, s)$ is defined by

$$\mathcal{G}_{r,s}(a_i, b_j; q, z) := z\,{}_r\Psi_s(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q, z). \qquad (1.6)$$

In this paper, we define the new operator, a q-analogue of the operator defined by Selvaraj and Karthikeyan [18] as follows:
$\mathcal{J}_\lambda^m(a_1, b_1; q, z)f : \mathcal{U} \longrightarrow \mathcal{U}$ by

$$\mathcal{J}_\lambda^0(a_1, b_1; q, z)f(z) = f(z) * \mathcal{G}_{r,s}(a_i, b_j; q, z).$$

$$\mathcal{J}_\lambda^1(a_1, b_1; q, z)f(z) = (1-\lambda)(f(z)*\mathcal{G}_{r,s}(a_i, b_j; q, z)) + \lambda\, z D_q(f(z)*\mathcal{G}_{r,s}(a_i, b_j; q, z)). \qquad (1.7)$$

$$\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z) = \mathcal{J}_\lambda^1(\mathcal{J}_\lambda^{m-1}(a_1, b_1; q, z)f(z)). \qquad (1.8)$$

If $f \in \mathcal{A}$, then from (1.7) and (1.8) we may easily deduce that

$$\mathcal{J}_\lambda^m(a_1, b_1; q, z)f = z + \sum_{n=2}^{\infty} [1 - \lambda + [n]_q\lambda]^m \; \Gamma_n a_n z^n, \qquad (1.9)$$

$$(m \in N_0 = N \cup \{0\} \text{ and } \lambda \geq 0)\,,$$

where

$$\Gamma_n = \frac{(a_1; q)_{n-1}(a_2; q)_{n-1} \ldots (a_r; q)_{n-1}}{(q; q)_{n-1}(b_1; q)_{n-1} \ldots (b_s; q)_{n-1}}, \quad (|q| < 1)\,.$$

Remark 1.1. *In this remark we list some special cases of the operator* $\mathcal{J}_\lambda^m(a_1, b_1; q, z)f$.

1. *For a choice of the parameter* $m = 0$*, the operator* $\mathcal{J}_\lambda^0(\alpha_1, \beta_1)f(z)$ *reduces to the q-analogue of Dziok- Srivastava operator [1].*

2. *For* $a_i = q^{\alpha_i}$, $b_j = q^{\beta_j}$, $\alpha_i, \beta_j \in \mathbb{C}, \beta_j \neq 0, 1, 2, \ldots, (i = 1, \ldots, r, j = 1, \ldots, s)$ *and* $q \to 1^-$*, we get the operator defined by Selvaraj and Karthikeyan [18].*

3. *For* $r = 2$, $s = 1$; $a_1 = b_1$, $a_2 = q$, *and* $\lambda = 1$*, we get the q- analogue of the well known Sălăgean operator (see [9, 17]).*

In addition, many integral and differential operators (both well known and new) can be obtained by specializing the parameters (see [1] and [18]) and references cited therein.

Let $\mathcal{S}_{q,t}^{\lambda,m}(a_1,\, b_1;\, A,\, B)$, $|t| \leq 1$, $t \neq 0$ denote the class of functions $f(z) \in \mathcal{A}$ satisfying the conditions

$$\frac{tz^2[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f(z)]'}{[\mathcal{J}_\lambda^1(a_1, b_1;\, q,\, z)g(z)][\mathcal{J}_\lambda^1(a_1, b_1;\, q,\, z)g(tz)]} \prec \frac{1+Az}{1+Bz}, \tag{1.10}$$

$$(-1 \leq B < A \leq 1, \quad z \in \mathcal{U}),$$

where $\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)g(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$.

Remark 1.2. *The following observations are obvious:*

(i) *If we let* $m = 0$, $r = 2$, $s = 1$, $a_1 = b_1$, $b_2 = 1$ *and* $q \to 1^-$, *then the class* $\mathcal{S}_{q,t}^{\lambda,m}(a_1,\, b_1;\, A,\, B)$ *reduces to* $X_t(A,\, B)$ *introduced and studied by [20].*

(ii) *If we let* $A = 1 - 2\alpha$, $B = -1$, $m = 0$, $r = 2$, $s = 1$, $a_1 = b_1$, $b_2 = 1$ *and* $q \to 1^-$, *then the class* $\mathcal{S}_{q,t}^{\lambda,m}(a_1,\, b_1;\, A,\, B)$ *reduces to* $X_t(\alpha)$ *introduced and studied in [19].*

Further, it can be easily seen that most of the well-known classes of analytic functions can be obtained as special cases of $\mathcal{S}_{q,t}^{\lambda,m}(a_1,\, b_1;\, A,\, B)$ *by specializing the parameters involved.*

By definition of subordination, equation (1.10) can be written as

$$\frac{tz^2[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f(z)]'}{[\mathcal{J}_\lambda^1(a_1, b_1;\, q,\, z)g(z)][\mathcal{J}_\lambda^1(a_1, b_1;\, q,\, z)g(tz)]} = \frac{1+Aw(z)}{1+Bw(z)}, \tag{1.11}$$

$$(\,-1 \leq B < A \leq 1,\, z \in \mathcal{U}),$$

where $w(z)$ is analytic in $\mathcal{U}$ and $w(0) = 0$, $|w(z)| < 1$. In the present work, we obtain the coefficient estimates, inclusion relation, distortion theorems, radius of convexity and Fekete-Szegö problem for the functions in the class $\mathcal{S}_{q,t}^{\lambda,m}(a_1,\, b_1;\, A,\, B)$. Our results extend some known results. Throughout our paper we use $-1 \leq B < A \leq 1$, $0 <| t |\leq 1$, $z \in \mathcal{U}$.

2 Main Results

2.1 Estimates for Coefficients

To prove the results in this subsection, we make use of the following Lemmas.

Lemma 2.1. *[15] Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be analytic in $\mathcal{U}$ and let $g(z) = \sum_{n=1}^{\infty} b_z z^n$ be analytic and convex in $\mathcal{U}$. If $f(z) \prec g(z)$, then $|a_n| \le |b_1|$, for $n = 1, 2, \ldots$.*

Lemma 2.2. *[21] Let $g(z) \in \mathcal{S}^*(\frac{1}{2})$ and $0 < |t| \le 1$, then $\frac{g(z)g(tz)}{tz} \in \mathcal{S}^*$.*

Using Lemma 2.2 and following the same steps as in [21], we get the following result.

Lemma 2.3. *Let $\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)g \in \mathcal{S}^*\left(\frac{1}{2}\right)$, then*

$$G(z) = \frac{[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)g(tz)]}{tz} = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{S}^*, \quad \text{and} \quad |c_n| \le n, \tag{2.1}$$

where $c_n = [\Omega_n b_n + \Omega_{n-1}\Omega_2 b_{n-1} b_2 t + \cdots + \Omega_n b_n t^{n-1}]$ and $\Omega_n = [1 - \lambda + [n]_q \lambda]^m\, \Gamma_n$.

Now we begin with following result.

Theorem 2.4. *If $f(z) \in \mathcal{S}_{q,t}^{\lambda, m}(a_1,\, b_1;\, A,\, B)$, then*

$$| a_n | \le \frac{1}{|\Gamma_n|[1 - \lambda + [n]_q \lambda]^m} \left(1 + \frac{(A - B)(n - 1)}{2}\right). \tag{2.2}$$

Proof. Let $f(z) \in \mathcal{S}_{q,t}^{\lambda, m}(a_1,\, b_1;\, A,\, B)$. Then there exists a function $p(z) \in \mathcal{P}$ analytic $\mathcal{U}$ such that

$$p(z) = \frac{z[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f(z)]'}{G(z)} \prec \frac{1 + Az}{1 + Bz} = 1 + (A - B)z + \cdots. \tag{2.3}$$

Clearly $\frac{1+Az}{1+Bz}$ is analytic and maps $\mathcal{U}$ into a convex domain. By Lemma 2.1,

$$|p_n| \le (A - B). \tag{2.4}$$

From (2.3), we have

$$\frac{z[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f(z)]'}{G(z)} = p(z), \quad (p(z) \in \mathcal{P}) \tag{2.5}$$

Using Lemma(2.3) in (2.5), we get

$$z + \sum_{n=2}^{\infty} [1 - \lambda + [n]_q \lambda]^m n \Gamma_n a_n z^n = (1 + \sum_{n=1}^{\infty} p_n z^n)(z + \sum_{n=2}^{\infty} c_n z^n) \tag{2.6}$$

Equating the coefficients of z^n in (2.6), we get

$$n\Gamma_n a_n[1-\lambda+[n]_q\lambda]^m = c_n + p_1c_{n-1} + p_2c_{n-2} + \cdots + p_{n-1}. \tag{2.7}$$

Therefore using (2.4) and Lemma 2.3 , we get

$$n\Gamma_n[1-\lambda+[n]_q\lambda]^m \mid a_n \mid \leq n\left(1+\frac{(A-B)(n-1)}{2}\right). \tag{2.8}$$

From (2.8), we easily obtain (2.2). This completes the proof. □

If we let $m=0$, $r=2$, $s=1$, $a_1=b_1$, $b_2=1$ and $q\to 1^-$ in Theorem 2.4, we get the following result

Corollary 2.5. *[20] If $X_t\,(A,\,B)$, then*

$$|a_n| \leq 1+\frac{(n-1)(A-B)}{2}.$$

Letting $A=1-2\alpha$, $B=-1$ in Corollary 2.5, the following result, due to Prajapat[9], becomes obvious.

Corollary 2.6. *If $f\in X_t\,(\alpha)$, then*

$$\mid a_n \mid\leq 1+(n-1)(1-\alpha).$$

2.2 Fekete-Szegö Problem.

We use the following lemmas to prove the results in this subsection:

Lemma 2.7. *[10] If $p(z)=1+p_1z+p_2z^2+\cdots$ is a function with positive real part, then for each complex number μ*

$$\mid p_2-\mu p_1^2 \mid\leq 2\max\left(1,\mid 2\mu-1\mid\right) \tag{2.9}$$

and the result is sharp for the functions given by

$$p(z)=\frac{1+z^2}{1-z^2},\quad p(z)=\frac{1+z}{1-z}.$$

Lemma 2.8. *[11] If*

$$G(z)=z+\sum_{n=2}^{\infty}c_nz^n\in\mathcal{S}^*,$$

then for each complex number λ *we have* $\mid c_3 - \lambda c_2^2 \mid \leq \max(1, \mid 3 - 4\lambda \mid)$ *and the result is sharp for the Koebe functon* $k(z)$ *if*

$$\left|\lambda - \frac{3}{4}\right| \geq \frac{1}{4}$$

and for

$$k^{\frac{1}{2}}\left(z^2\right) = \frac{z}{1-z^2}$$

if

$$\mid \lambda - \frac{3}{4} \mid \leq \frac{1}{4}.$$

Theorem 2.9. *If* $f(z) \in \mathcal{S}_{q,\,t}^{\lambda,\,m}(a_1,\, b_1;\, A,\, B)$*, then for* $\mu \in \mathbb{C}$ *we have*

$$\mid a_3 - \mu a_2^2 \mid \leq \frac{A-B}{3\gamma_3[1-\lambda+[3]_q\lambda]^m} \max(1, \mid 2\beta - 1 \mid) + \frac{1}{3}\max(1, \mid 3 - 4\mu_1 \mid) + \frac{2(A-B)}{\gamma_3[1-\lambda+[3]_q\lambda]^m}\left|\frac{1}{3} - \frac{\mu}{2}\right|, \tag{2.10}$$

where

$$\beta = \frac{1+B}{2} + \frac{3(A-B)\mu\Gamma_3[1-\lambda+[3]_q\lambda]^m}{8\Gamma_2^2[1-\lambda+[2]_q\lambda]^{2m}},\ \mu_1 = \frac{3\mu}{4}.$$

Proof. As $f(z) \in \mathcal{S}_{q,\,t}^{\lambda,\,m}(a_1,\, b_1;\, A,\, B)$, by (1.11) we have

$$\frac{z[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f(z)]'}{G(z)} = \frac{1+Aw(z)}{1+Bw(z)}.$$

Let

$$h(z) = \frac{1+w(z)}{1-w(z)} = 1 + p_1 z + p_2 z^2 + \cdots.$$

Then $Re(h(z)) > 0$ and $h(0) = 1$. Hence,

$$\frac{z[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f(z)]'}{G(z)} = \frac{1-A+h(z)(1+A)}{1-B+h(z)(1+B)}. \tag{2.11}$$

From (2.11), we obtain

$$1 + \gamma_2[1-\lambda+[2]_q\lambda]^m(2a_2 - c_2)z + \gamma_3[1-\lambda+[3]_q\lambda]^m(3a_3 - 2a_2c_2 - c_3 + c_2^2)z^2 + \cdots = 1 + \frac{p_1(A-B)}{2}z + \frac{(A-B)}{2}\left(p_2 - p_1^2(\frac{1+B}{2})\right)z^2 + \cdots. \tag{2.12}$$

Equating the coefficients at z and z^2 on both sides of the above equation, we get

$$a_2 = \frac{p_1(A-B) + 2c_2\gamma_2[1-\lambda+[2]_q\lambda]^m}{4\gamma_2[1-\lambda+[2]_q\lambda]^m}$$

and

$$a_3 = \frac{1}{3}\left[c_3 + \frac{(A-B)}{2\gamma_3[1-\lambda+[3]_q\lambda]^m}\left(p_1c_2 + p_2 - \frac{p_1^2(1+B)}{2}\right)\right].$$

Therefore, we have

$$\begin{aligned} \mid a_3 - \mu a_2^2 \mid \leq & \frac{A-B}{6\gamma_3[1-\lambda+[3]_q\lambda]^m} \mid p_2 - \beta p_1^2 \mid + \frac{\mid c_3 - \mu_1 c_2^2 \mid}{3} \\ & + \frac{(A-B)}{2\gamma_3[1-\lambda+[3]_q\lambda]^m} \mid c_2 \mid (\frac{1}{3} - \frac{\mu}{2}) \mid p_1 \mid . \end{aligned} \tag{2.13}$$

Using Lemmas 2.7 and 2.8, the proof is complete.

□

If we let $m = 0$, $r = 2$, $s = 1$, $a_1 = b_1$, $b_2 = 1$ and $q \to 1^-$ in Theorem2.4, we get the following result

Corollary 2.10. *[20] If $f(z) \in X_t\,(A,\, B)$, then for $\mu \in \mathbb{C}$ we have*

$$\begin{aligned} \mid a_3 - \mu a_2^2 \mid \leq & \frac{A-B}{3}\max(1, \mid 2\beta - 1 \mid) + \frac{1}{3}\max(1, \mid 3 - 4\mu_1 \mid) \\ & + 2(A-B)\left|\frac{1}{3} - \frac{\mu}{2}\right|, \end{aligned} \tag{2.14}$$

where

$$\beta = \frac{1+B}{2} + \frac{3(A-B)\mu}{8},\ \mu_1 = \frac{3\mu}{4}.$$

For $A = 1-2\alpha, B = -1$, Theorem 2.9 gives the following result.

Corollary 2.11. *If $\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f(z) \in X_t(\alpha)$, then for $\mu \in \mathbb{C}$,*

$$\begin{aligned} \mid a_3 - \mu a_2^2 \mid \leq & \frac{2(1-\alpha)}{3\gamma_3[1-\lambda+[3]_q\lambda]^m}\max(1, \mid 2\beta - 1 \mid) + \frac{1}{3}\max(1, \mid 3 - 4\mu_1 \mid) \\ & + \frac{4(1-\alpha)}{\gamma_3[1-\lambda+[3]_q\lambda]^m}\left|\frac{1}{3} - \frac{\mu}{2}\right|, \end{aligned}$$

where

$$\beta = \frac{3(1-\alpha)\mu\Gamma_3[1-\lambda+[3]_q\lambda]^m}{4\Gamma_2^2[1-\lambda+[2]_q\lambda]^{2m}},\ \mu_1 = \frac{3\mu}{4}.$$

2.3 Inclusion relation

The following lemma is useful in the proof of the main result in this subsection.

Lemma 2.12. *If* $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1,$ *then*

$$\frac{1 + A_1 z}{1 + B_1 z} \prec \frac{1 + A_2 z}{1 + B_2 z}.$$

Theorem 2.13. *Let* $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1.$ *Then*

$$\mathcal{S}^{\lambda,\, m}_{q,\, t}(a_1,\, b_1;\, A_1,\, B_1) \subset \mathcal{S}^{\lambda,\, m}_{q,\, t}(a_1,\, b_1;\, A_2,\, B_2).$$

Proof. As $f(z) \in \mathcal{S}^{\lambda,\, m}_{q,\, t}(a_1,\, b_1;\, A_1,\, B_1),$

$$\frac{tz^2[\mathcal{J}^m_\lambda(a_1, b_1;\, q,\, z)f(z)]'}{[\mathcal{J}^m_\lambda(a_1, b_1;\, q,\, z)g(z)]g(tz)} \prec \frac{1 + A_1(z)}{1 + B_1(z)}. \tag{2.15}$$

Since $-1 \leq B_2 \leq B_1 < A_1 < A_2 \leq 1,$ by Lemma 2.12, we have

$$\frac{tz^2[\mathcal{J}^m_\lambda(a_1, b_1;\, q,\, z)f(z)]'}{[\mathcal{J}^m_\lambda(a_1, b_1;\, q,\, z)g(z)]g(tz)} \prec \frac{1 + A_1(z)}{1 + B_1(z)} \prec \frac{1 + A_2(z)}{1 + B_2(z)}. \tag{2.16}$$

This yields $f(z) \in \mathcal{S}^{\lambda,\, m}_{q,\, t}(a_1,\, b_1;\, A_2,\, B_2)$ and this proves the inclusion relation. □

2.4 Distortion theorems

Theorem 2.14. *If* $f(z) \in \mathcal{S}^{\lambda,\, m}_{q,\, t}(a_1,\, b_1;\, A,\, B),$ *then for* $\mid z \mid = r,\ 0 < r < 1,$ *we have*

$$\frac{(1 - Ar)}{(1 - Br)(1 + r)^2} \leq \mid [\mathcal{J}^m_\lambda(a_1, b_1;\, q,\, z)f(z)]' \mid \leq \frac{(1 + Ar)}{(1 + Br)(1 - r)^2} \tag{2.17}$$

and

$$\int_0^r \frac{(1 - At)}{(1 - Bt)(1 + t)^2} dt \leq \mid \mathcal{J}^m_\lambda(a_1, b_1;\, q,\, z)f(z) \mid \leq \int_0^r \frac{(1 + At)}{(1 + Bt)(1 - t)^2} dt \tag{2.18}$$

Proof. From (2.4), we have

$$|[\mathcal{J}^m_\lambda(a_1, b_1;\, q,\, z)f(z)]'| = \frac{|G(z)|}{\mid z \mid} \left| \frac{1 + Aw(z)}{1 + Bw(z)} \right|,\ w(z) \in \mathcal{U}. \tag{2.19}$$

It is easy to show that the transform

$$\frac{z[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f(z)]'}{G(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}$$

maps $\mid w(z) \mid \leq r$ onto the circle

$$\left| \frac{z[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f(z)]'}{G(z)} - \frac{1 - ABr^2}{1 - B^2r^2} \right| \leq \frac{(A - B)r}{1 - B^2r^2}, \quad \mid z \mid = r. \tag{2.20}$$

This implies that

$$\frac{1 - Ar}{1 - Br} \leq \left| \frac{1 + Aw(z)}{1 + Bw(z)} \right| \leq \frac{1 + Ar}{1 + Br}. \tag{2.21}$$

Since by Lemma 2.3, $G(z)$ is a starlike function and so due to a well known result, we have

$$\frac{r}{(1 + r)^2} \leq \mid G(z) \mid \leq \frac{r}{(1 - r)^2}. \tag{2.22}$$

Equation (2.19) together with (2.21) and (2.22) yield (2.17). On integrating (2.17) from 0 to r, (2.18) follows. □

If we let $A = 1 - 2\alpha$, $B = -1$ in Theorem 2.14 gives the following result similar to the result obtained by Prajapat[19]:

Corollary 2.15. *If* $[\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f(z)] \in X_t(\alpha)$, *then*

$$\frac{(1 - (1 - 2\alpha)r)}{(1 + r)^3} \leq \mid [\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f(z)]' \mid \leq \frac{(1 + (1 - 2\alpha)r)}{(1 - r)^3} \tag{2.23}$$

and

$$\int_0^r \frac{(1 - (1 - 2\alpha)t)}{(1 + t)^3} dt \leq \mid \mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f(z) \mid \leq \int_0^r \frac{(1 + (1 - 2\alpha)t)}{(1 - t)^3} dt. \tag{2.24}$$

2.5 Radius of convexity

Theorem 2.16. *If* $f(z) \in \mathcal{S}_{q,t}^{\lambda,m}(a_1,\, b_1;\, A,\, B)$, *then* $\mathcal{J}_\lambda^m(a_1, b_1;\, q,\, z)f(z)$ *is convex in* $\mid z \mid = r_1$, *where* r_1 *is the smallest positive root in* $(0, 1)$ *of the equation*

$$ABr^3 - A(B - 2)^2 - (2B - 1)r - 1 = 0. \tag{2.25}$$

Proof. As $f(z) \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, we have

$$z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]' = G(z)p(z). \tag{2.26}$$

By applying logarithmic differentiation in (2.26), we get

$$1 + \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]''}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'} = \frac{zG'(z)}{G(z)} + \frac{zp'(z)}{p(z)} \tag{2.27}$$

Now for $G(z) \in \mathcal{S}^*$, we have

$$Re\left(\frac{zG'(z)}{G(z)}\right) \geq \frac{1-r}{1+r}.$$

Therefore, (2.27) yields

$$Re\left(1 + \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]''}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}\right) \geq \frac{1-r}{1+r} - \left|\frac{zp'(z)}{p(z)}\right|.$$

Further, we have

$$Re\left(1 + \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]''}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}\right) \geq \frac{1-r}{1+r} - \frac{r(A-B)}{(1+Ar)(1+Br)}.$$

By a straightforward computation, we have

$$Re\left(1 + \frac{z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]''}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}\right) \geq \frac{-ABr^3 + A(B-2)r^2 + (2B-1)r + 1}{(1+r)(1+Ar)(1+Br)}.$$

Hence, the function $\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)$ is convex in $|\, z \,| < r_1$, where r_1 is the smallest positive root in (0, 1) of the equation

$$ABr^3 - A(B-2)r^2 - (2B-1)r - 1 = 0.$$

□

Letting $A = 1 - 2\alpha$, $B = -1$ in Theorem 2.16 gives the following result which is similar to the result obtained by Prajapat[19]:

Corollary 2.17. *If* $f(z) \in X_t(\alpha)$, *then* $\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)$ *is convex in* $|\, z \,| < r_0 = 2 - \sqrt{3}$.

Conclusion. The motivation of the class $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$ was from the study of univalent functions with respect to symmetric points. Many authors introduced and defined various classes of analytic functions with respect to symmetric points. For appropriate choice of parameters involved in the class $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, several results follow as special case of our results.

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