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# On Maximal Cycles or Triangular Planar Polygonal Graphs and Their Coloring

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**Abstract**

We analyze the cycles or planar polygonal graphs G which are maximal in their inner edges and offer a series of coloring results, such as $\chi(G) = 3$ or $P(G, 3) = 6$, or construction algorithms, among others. Some aspects of them with various applications in path modeling, data flow design, computer networks or best resource allocation are discussed.

# 1 Introduction

In this study, we will consider a type of planar graphs, of polygonal structure, the so-called cycles. We will show some results referring to their vertices coloration when they are triangular or maximal in their inner edges [1][2][3][4].

We will begin by recalling the following theorem which is fundamental in our study:

**Theorem 1.1.** *In a maximal or triangular planar graph $G$, $E = 3V - 6$.*

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**Proof.** From Euler's formula, $F + V = E + 2$, $V$ being the number of vertices, $E$ the number of edges, and $F$ the number of faces. It is well known that $E \leq 3V - 6$ in a planar graph in which $V \geq 3$. In the case of a maximal or triangular planar graph, we have $E = 3V - 6$. ■

For clarity, we have the following definitions:

**Definition 1.2.** *Let $n \geq 3$. A Cycle or polygonal graph is a graph $G$ with $n$ vertices $V_1$ to $V_n$ and a perimeter denoted by $\overline{V_1V_2}$, $\overline{V_2V_3}$,..., $\overline{V_{n-1}V_n}$, $\overline{V_nV_1}$. This perimeter defines a topological boundary between the exterior and the interior of the polygonal shape.*

**Definition 1.3.** *A planar polygonal maximal or triangular graph is the polygonal planar graph to which all interior and exterior edges are added.*

**Definition 1.4.** *As a result, we can talk about the exterior ($V_E$), interior ($V_I$) and perimeter ($V_P$) so that $V = V_E + V_I + V_P$. Similarly, we can talk about the external ($E_E$), interior ($E_I$) and perimeter ($E_P$) so that $E = E_E + E_I + E_P$.*

## 2 Some properties of polygonal planar graphs

We mention some theorems that will help characterize polygonal planar graphs:

**Theorem 2.1.** *In a maximal or triangular polygonal planar graph, without exterior or interior vertices, $E_I = E_E = V_P - 3$.*

**Proof.** Since $E = 3V - 6$, $E_E + E_I + E_P = 3(V_E + V_I + V_P) - 6$.

We have $V_E = 0$ and $V_I = 0$. On the other hand, the perimeter of vertices and edges leads to $E_P = V_P$.

Moreover, considering the graph topologically, due to the geometric inversion between the interior and the exterior,the perimeter, the various interior edges between vertices can be drawn in the same way (inverted) on the outside of the perimeter border, having as many edges inside as outside. If they do not cross in the interior, they will not do so in the exterior, and if they cross in the interior they would also do so in the exterior, starting and ending at the same vertex. So there will be as many edges on the outside as on the inside in a maximal graph. Consequently, $E_E = E_I$.

As a result, $2E_I + V_P = 3(V_P) - 6$. Finally, $E_I = E_E = V_P - 3$. ■

**Theorem 2.2.** *In a planar maximal polygonal graph or triangular graph without outside vertices,* $E_I = 3V_I + V_P - 3$.

**Proof.** Since $E = 3V - 6$ and distinguishing edges and vertices as expressed in the preceding definitions, we have that $E_E + E_I + E_P = 3(V_E + V_I + V_P) - 6$.

First $V_E = 0$. Next, the perimeter forces $E_P = V_P$. As in the preceding theorem 2.1, $E_E = V_P - 3$. Consequently, $E_I = 3V_I + V_P - 3$. ■

**Theorem 2.3.** *In a planar graph polygonal maximal or triangular where there are no outer edges,* $E_E = 0$. *The ratio of interior vertices and interior edges is given by* $V_I = (V_P - 3) - \lambda$ *and* $E_I = 4(V_P - 3) - 3\lambda$, *with* $\lambda \leq V_P - 3$, $\lambda \in \mathbb{Z}$.

**Proof.** From $E_I = V_P + 3V_I - 3$, we have $E_I - 3V_I = V_P - 3$ which is a linear Diophantine equation of the type $ax + by = n$ which can be solved, by defining $d = gcd(a, b)$. If $d|n$, then there is a solution. In addition, $d = \alpha a + \beta b$, being a particular solution $x_0 = \frac{n}{d}\alpha$, $y_0 = \frac{n}{d}\beta$. AMoreover, the remaining solutions are $x = x_0 + \frac{b}{d}\lambda$, $y = y_0 - \frac{a}{d}\lambda$.

In our case, $d = gcd(1, -3) = 1$. So there is always a solution. It turns out that $1 = \alpha - 3\beta$. Consider the values $\alpha = 4$ and $\beta = 1$. A particular solution is $(E_{I_0}, V_{I_0}) = (4(V_P - 3), (V_P - 3))$. The possible solutions are $(E_I, V_I) = (4(V_P - 3) - 3\lambda, (V_P - 3) - \lambda)$, with $\lambda \leq V_P - 3$. ■

**Theorem 2.4.** *In a maximal or triangular planar polygonal graph, only* 3 *vertices are accessible from the outside of it.*

**Proof.** In the case of having more than 3 vertices, the outer perimeter will force having two non-consecutive vertices and drawing an outer edge between them. This leaves the vertices existing between them in the inner part of the edge not being accessible from outside the polygon.

We could build a new figure with a smaller number of exterior vertices until no more edges can be drawn; that is, when $V_P - 3 = E_E = 0$. So there will be 3 vertices which are those that no longer allow drawing more edges (all of them being interconnected with each other) being the only vertices that are totally outside a polygonal planar maximal graph. ■

**Theorem 2.5.** *In a planar maximal or triangular polygonal graph, where there are neither outer edges ($E_E = 0$) nor inner vertices ($V_I = 0$), the maximum number of vertices that are not attached together but have edges with all the vertices of the graph is $\lceil \frac{V-1}{2} \rceil$, and the minimum number is* 1.

**Proof.** First, consider $E_I = 0$. Since there are no more than these, the various perimeter vertices $V_P$, joined with its predecessor and its successor in a modular way from 1 to $n$, the total number will be $\lceil \frac{V-1}{2} \rceil$.

However, we must place the inner edges, a total of $E_I = V - 3$, which at least in one of the possible placements we have one that joins one of the vrtices, for example $V_n$, with the rest, from $V_2$ to $V_{n-2}$.

Taking the set of odd vertices $\{V_1, V_3, ...\}$ to $V_{n-1}$, we have a set of vertices not attached to each other. The same will happen with the set of even vertices $\{V_2, V_4, ...\}$ to $V_{n-1}$. If $n$ is odd, both sets (the one of even vertices and the one of odd vertices), have cardinality equal to the maximum value $\lceil \frac{V-1}{2} \rceil$, but if $n$ is even, only the set of odd vertices will be maximum.

In any case, this maximum value $\lceil \frac{V-1}{2} \rceil$ can always exist in any maximal polygonal graph with this edge configuration.

In the case of the minimum, it is clear that if the vertex $V_n$ is taken, attached to all the others, only this vertex will form a set of vertices that will be joined to the rest, so its cardinality is unity. ■

# 3 Some coloring theorems

We will begin this section with a procedure, which will be used to select a set of vertices not connected to each other in a polygonal maximal graph without outside edges.

Later we will see the importance of this particular selection.

**Algorithm.**
First,we arrange the graph $G$ in the form of a linear sequence of vertices with a single vertex $V_k$ at the bottom, following a triangular shape, as it appears in Fig. 1.

To construct the set of vertices $\gamma$, we start by taking as a first vertex the lower vertex of our triangular arrangement, $V_k$.

This choice requires that the successive and preceding vertices of $V_k$ cannot belong to the set $\gamma$, since that set contains vertices not attached to each other. Thus, $V_{k-1}$ and $V_{k+1} \notin \gamma$. Similarly, this happens with any other vertices to which $V_k$ is attached. Also each of them divides the horizontal sequence of vertices into fragments, subgraphs, which must be analyzed separately.

Suppose, for example, we have two vertices that delimit one of these sections, say $V_m$ and $V_n$. Due to the maximality of edges, $V_m$ and $V_n$ will be joined by an edge which subtends from vertex $V_k$, forming a triangle using these three vertices.

The algorithm will continue analyzing the rest of the vertices and edges, the subgraph inside this edge $\overline{V_mV_n}$.

Thus, if we removed the edge $\overline{V_mV_n}$ we would have a single vertex $V_i$ now in the horizontal line of vertices that would be accessible, due to the formation of triangles, specifically that formed by $V_i$, the edge $\overline{V_mV_n}$, and the vertices $V_m$ and $V_n$. Hence there is only one vertex $V_i$. This vertex $V_i$, being under the edge $\overline{V_mV_n}$ has no connection to the vertex $V_k \in \gamma$, so it becomes part of that set.

When taking the value $V_i$ for the set of vertices $\gamma$, we will have to eliminate its edges which will indicate another series of vertices to which it is attached that cannot be part of $\gamma$. The vertices which are not included will have generated new subsections in the horizontal line of vertices that must be analyzed taking two consecutive vertices that have been excluded from the set $\gamma$ and looking under the edge that joins them, to take a new vertex that we will include in $\gamma$. The process continues as such .

The process comes to an end because at each step you look under an edge and an additional vertex is taken for the $\gamma$ set, getting to analyze all the subgraphs or sections generated by the vertices not included in that set.

**Example.** Consider a graph $G$ as in Fig. 1, with 18 vertices, planar polygonal graph maximal in its interior edges. We have it in the form offered in Fig. 1, with one of its vertices, $V_{18}$, at the bottom, and the rest aligned at the top.

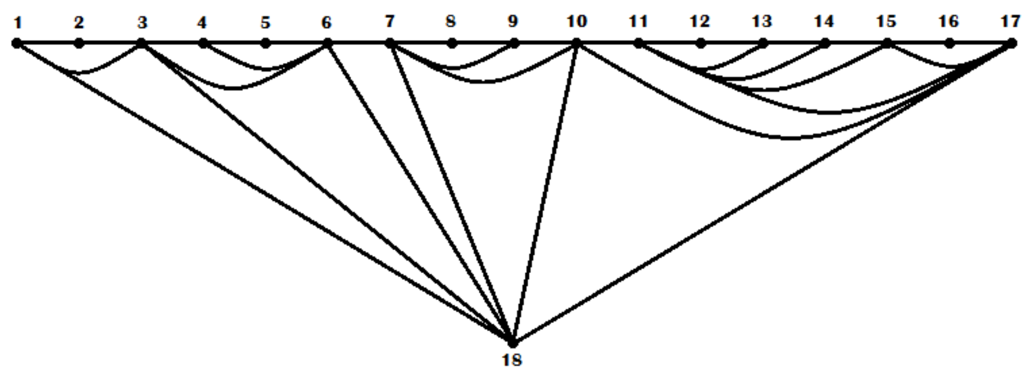


Figure 1: Graph example where we will apply the vertex selection algorithm.

You start by taking the vertex $V_{18}$ in the set $\gamma$. The edges that start from $V_{18}$ are connected with vertices that cannot belong to $\gamma$. They are $V_1$, $V_3$, $V_6$, $V_7$, $V_{10}$ and $V_{17}$. They determine five zones or subgraphs to be analyzed with the same procedure.

Starting with the area bounded by $V_1$ and $V_3$, the edge that joins them, $\overline{V_1V_3}$, shows inside an accessible vertex that is not attached to any point of the set $\gamma$. With this, $V_2 \in \gamma$. Its edges will indicate vertices that cannot belong to $\gamma$. In this subgraph the process is over.

In the following subgraph, bounded by $V_3$ and $V_6$, looking at the edge $\overline{V_3V_6}$ we find the vertex $V_4$, which we can incorporate into $\gamma$. Its edges indicate vertices that cannot be incorporated into our set. This also ends the analysis in this subgraph.

In the subgraph bounded by $V_6$ and $V_7$, there are no more vertices to consider. So we move on to the subgraph bounded by $V_7$ and $V_{10}$. Looking at the vertices inside the edge that joins these vrtices, $\overline{V_7V_{10}}$, we find the vertex $V_9$, which cannot not be attached to any other vertex of the set $\gamma$. So it is selected, $V_9 \in \gamma$. Its edges will give us more vertices that we cannot take for $\gamma$ because they are attached to it. Not being able to continue after having exhausted all the vertices, we are done with this subgraph.

Considering the section of the graph bounded by $V_{10}$ and $V_{17}$ we have to look under its edge $\overline{V_{10}V_{17}}$, where we find one vertex. So $V_{11} \in \gamma$. Its edges connect with another sequence of vertices that cannot be assumed as elements of $\gamma$. Another zone bounded by vertices $V_{15}$ and $V_{17}$ emerged in the process. The analysis of this zone when looking under the edge $\overline{V_{15}V_{17}}$ allows us to take the vertex $V_{16}$ for $\gamma$.

Thus, all the areas bounded by vertices that cannot belong to $\gamma$ have been analyzed. The process is over, and we have $\gamma = \{V_2, V_4, V_9, V_{11}, V_{16}, V_{18}\}$, see Fig. 2.

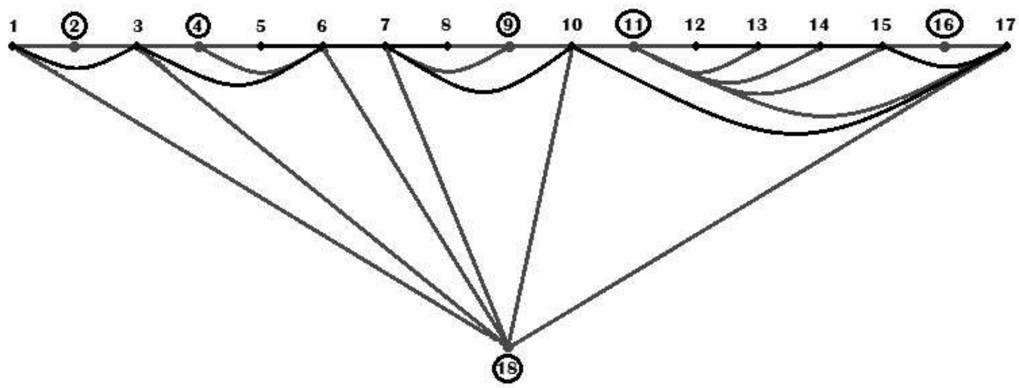


Figure 2: Graph resolved after application of the vertex selection algorithm.

**Theorem 3.1.** *Every maximal polygonal planar graph $G$ without outside edges can be divided into two subgraphs: one of them, $G_1$ is a tree, and the other, $G_2$ is an absolutely non-connected subgraph, a set of vertices, with their respective edges, not attached between them.*

**Proof.** From the indicated algorithm it is clear that the set $\gamma$ is a set of vertices not attached to each other. Taking the first one, its edges are observed so as not to take any vertex attached to it, and so on in the whole process. On the other hand, by the described process there are no more vertices that we can include in the $\gamma$ set once the algorithm is finished; that is, the rest of the vertices of the initial graph $G$ are attached to some vertices of $\gamma$. Therefore, $G_2$ is a subgraph of $G$ formed by vertices not attached to each other, absolutely non-connected, together with their respective edges.

On the other hand, if we look at subgraph $G_1$, all its vertices are linked together. Assuming an isolated vertex of $G_1$ was given, not attached to any other of $G_1$. There would be at least one triangulation between the said vertex and two others, it would be the case that those other two vertices will belong to $G_2$; that is, they were chosen for $\gamma$. The vertices of $\gamma$ are not attached between them. If they formed a triangulation at the beginning, they would be united. But this cannot be. Therefore, there are no vertices of subgraph $G_1$ not attached to another vertex of $G_1$.

In addition, between the vertices of $G_1$ there are no closed circuits, since if there were, we could continue with the described algorithm, calculating under the said edge that encloses the vertex or interior vertices, and there will always be some that can be considered members of $\gamma$, breaking under the said vertices and their edges, the closed circuit. With this, any original triangulation will be distributed between vertices of the graphs $G_1$ and $G_2$.

With all this, $G_1$ is a tree. ■

**Corollary 3.2.** *Every maximal planar polygonal graph $G$ without outside edges is 3-colorable, $\chi(G) = 3$.*

By theorem 3.1, every maximal polygonal graph $G$ without outside edges can be divided into a subgraph $G_1$ which is a tree, and another subgraph $G_2$ that is constituted by vertices not attached to each other. So we have to give a color to the vertices of $G_2$ and apply on $G_1$, a tree, two different colors from the color given to $G_2$. ■

**Theorem 3.3.** *Every planar maximal polygonal graph $G$ without outside edges, which we know can be divided into two subgraphs, one of them a tree and the other a set of vertices with their edges but not attached together, absolutely non-connected graph, can be divided according in this manner, into three different ways.*

**Proof.** By the described algorithm, one can start with any of the vertices. We will assume that $G$ had $n$ vertices.

If the algorithm is started at a certain vertex $V_i$ and a set $\gamma$ is obtained, which we will call $\gamma_1$, by the constructive process of the procedure, it would be the same to start with any of the vertices that make up the set $\gamma_1$ since the result would be the same, both for the tree subgraph obtained as for the set of vertices of $\gamma_1$. This is because that vertex with which the algorithm starts from the edges attached to it will begin to consider vertices attached to it that cannot be part of $\gamma_1$ and to bound subzones, which will be given by those vertices of the tree, being always those same ones for any vertices of $\gamma_1$ that we begin with.

The rest of the vertices with their edges form a tree subgraph, which can be divided into two sets of vertices, which we will call $\gamma_2$ and $\gamma_3$, each of which is formed by vertices not attached to each other, absolutely non-connected.

On the other hand, the graph $G$ is made up of several minor triangles, always attaching a set of three vertices. Since the vertices are linked together for each of these triangles because the various $\gamma_i$ are formed by vertices not attached to each other, it is clear that because these vertices of these triangles are joined, each of them must belong to a different set $\gamma_i$.

Hence, given a set of vertices $\gamma_2$ and its subgraph, the rest of vertices and edges are formed by vertices of $\gamma_1$ together with the vertices of $\gamma_3$, which are joined together in two of the vertices of the smaller triangles. They form a tree subgraph since there would not be a cycle either because that would mean that $\gamma_2$ would lack at least one element. Similarly for the set $\gamma_3$.

As a result, it is clear that there are three and only three possible configurations of the constitution of graph $G$ in subdivisions of two subgraphs, one of them a tree and the other a set of vertices not attached to each other, with their various edges. ∎

**Example.** Returning to the example shown above, we would have:

$$\gamma_1 = \{V_2, V_4, V_9, V_{11}, V_{16}, V_{18}\},$$
$$\gamma_2 = \{V_1, V_6, V_8, V_{10}, V_{13}, V_{15}\},$$
$$\gamma_3 = \{V_3, V_5, V_7, V_{12}, V_{14}, V_{17}\}.$$

**Corollary 3.4.** *If in a maximal polygonal graph $G$ of $n$ vertices without outside edges, of the three possibilities of the sets $\gamma_1$, $\gamma_2$ and $\gamma_3$, one of them has minimal cardinality, the cardinality of the other sets will be maximum if $n$ is odd, or if $n$ is even, the cardinality of one of them will be maximum and the other maximum-1, and vice versa.*

**Proof.** If for example $\#\gamma_1 = 1$, then in the construction algorithm we have that there is a vertex, the only one of $\gamma_1$. Suppose $V_i$ is attached to the rest of the vertices. Then, applying the algorithm by placing the vertex $V_{i+1}$ at the bottom of the triangle, we would take the various vertices $V_{i+3}$, $V_{i+5}$, $V_{i+7}$, ... $= \{V_{i+(2k+1)}\}$, for the set $\gamma_2$, $k \in \mathbb{N}$. Similarly, the set $\gamma_3$ will be the set of vertices $V_{i+2}$, $V_{i+4}$, $V_{i+6}$, ... $= \{V_{i+2k}\}$, with $k \in \mathbb{N}$. Since the maximum value is $\lceil \frac{V-1}{2} \rceil$, if $n$ is odd $\#\gamma_2 = \#\gamma_3 = \lceil \frac{V-1}{2} \rceil$, and if $n$ is even, $\#\gamma_2 = \lceil \frac{V-1}{2} \rceil$ and $\#\gamma_3 = \lceil \frac{V-1}{2} \rceil - 1$.

On the contrary, if $\#\gamma_2 = \lceil \frac{V-1}{2} \rceil$ it is because, in the configuration, the lower vertex is taken, which we will call $V_i$ and the sequence of all successive vertices with jump 2, $V_{i+2k}$, with $k \in \mathbb{N}$. In this configuration we will have the vertex $V_{i-1}$ attached to all the vertices. The latter means that this vertex $V_{i-1}$ will be the only element of a set $\gamma$ of cardinality one. If $G$ has an even set of vertices it is clear that the cardinality of the other set $\gamma$ will be the maximum minus 1, $\lceil \frac{V-1}{2} \rceil - 1$, and if it is odd it will be of maximum cardinality, since $\#\gamma_1 + \#\gamma_2 + \#\gamma_3 = n$. ■

**Corollary 3.5.** *A maximal polygonal graph $G$ without outside edges can be colored in three colors and in six different ways, resulting in its chromatic polynomial $P(G,0) = P(G,1) = P(G,2) = 0$ and $P(G,3) = 6$.*

**Proof.** We have seen that at least three colors are needed, $\chi(G) = 3$. On the other hand, the value of six is the simple result of calculating the permutations of the three colors. ■

# 4 Coloring adding outer edges and vertices

once the outer edges have been introduced though, it would not be correct to assume a 3-colourable polygonal graph.

We consider two different situations:

- On the one hand, we will show the simplest case (with fewer vertices) of the maximal planar polygonal graph with exterior edges where 4 colors are required for coloring (Fig. 3).

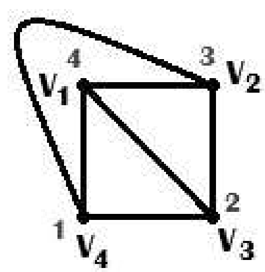


Figure 3: Simpler polygonal planar graph with outer edges and not 3-colorable.

- On the other hand, we will show the simplest planar polygonal graph (with fewer vertices, four) inscribed or circumscribed by the minimum number of vertices, two, where 4 colors are required for the coloring of the polygonal cycle or subgraph (Fig. 4).

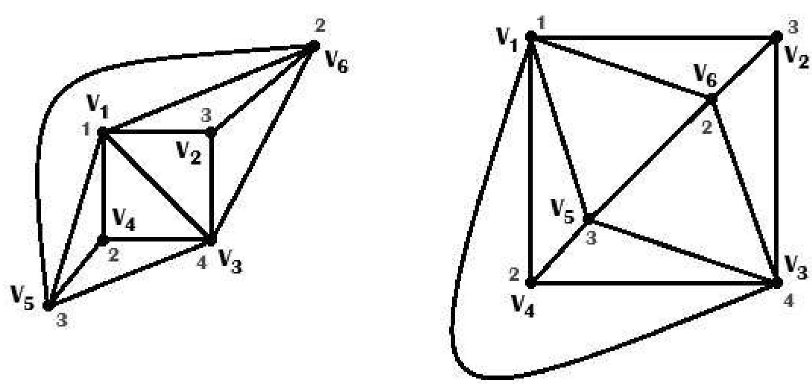


Figure 4: Simpler polygonal planar graphs with the least number of circumscribed or inscribed vertices, which prevent a 3-colouring of the polygonal subgraph.

## 5 Conclusions

In this study, we have considered the cycles or planar polygonal graphs, maximal or triangular inside, and we have given a few properties. An algorithm has been developed to decompose them into a tree-type subgraph and an absolutely non-connected subgraph. We have calculated the number of different separations in these two types of subgraphs that can be achieved, always being a total of three, which leads to a value in the chromatic polynomial $P(G, 3) = 6$.

In addition, the chromatic number has been calculated for this type of graphs, proving to be 3-colorable, and we have given some results on the cardinality in the number of vertices of the previous subgraphs.

As in the study of graphs, we believe that the results offered here can be useful in a variety of fields and situations. We briefly indicate the modeling of paths, optimal routes, construction of transport systems as well as the design of data flows in the computing sector, the structure of computer networks and social networks, or the design of any other perimeter structure (maximal planar graph) of polygonal shapes, or the like, as occurs in geographic territories, border perimeters of countries or regions, geographical boundaries by water, such as lakes, or even satellite orbits, etc., where we cannot locate central elements, and where there are point-to-point communications, among other similar examples, in which the union of all the vertices is necessary and certain properties are required, linked to the coloration, or to look for the tree subgraph and the absolutely non-connected subgraph of a graph, which allows to arrange and allocate in the best way a series of resources, data, merchandise, etc.

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# References

[1] Béla Bollobás, Graph Theory, Springer: New York, Berlin, 1979.

[2] Oleg V. Borodin, Colorings of plane graphs: A survey, Discrete Mathematics, **313,** no. 4, (2013), 517-539.

[3] Reinhard Diestel, Graph Theory, Springer: Berlin, Heidelberg, 2005.

[4] Sriram V. Pemmaraju, and Steven S. Skiena, Computational Discrete Mathematics: Combinatorics and Graph Theory in Mathematica, Cambridge University Press: Cambridge, 2003.