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On the edge irregularity strength of bipartite graph and corona product of two graphs

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Abstract

For a simple graph G, a map $\phi : V(G) \to \{1, 2, \ldots, k\}$ is called a vertex k-labeling. For any edge vu in G, its weight $\phi(vu) = \phi(v) + \phi(u)$. If all of the edges weights are distinct, then ϕ is called an edge irregular k-labeling of G. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, denoted by $es(G)$. In this paper, we determine an exact value of edge irregularity strength of complete bipartite graph $K_{n,2}$, corona product of P_n with P_6 and P_n with C_3.

1 Introduction

Let G be a simple graph and ϕ be a vertex k-labeling for G. Let $e = uv$ be an edge in $E(G)$, define the weight of e by $\phi(e) = \phi(uv) = \phi(u) + \phi(v)$. If all

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of the edges weights are distinct, then ϕ is called an edge irregular k-labeling of G. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, denoted by $es(G)$. Also, if the domain of ϕ is the edge set $E(G)$, such that $\phi(v_1v_2) = m \in \{1, 2, \ldots, k\}$, then ϕ is called the edge k-labeling. In particular, if $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$, then ϕ is called the total k-labeling map of G.

The graph labeling has wide range of applications, for instance X-ray, radar, circuit design, network and communication design. The most complete recent survey of graph labelings is [6].

In [3], the edge irregularity strength for some graphs is investigated, such as P_n, the star graph $K_{1,n}$, double star graph $S_{m,n}$ and Cartesian product of two paths P_n and P_m. In [4], Mushayt found the edge irregularity strength of Cartesian product of stars, cycle with path P_2 and strong product of path P_n with P_2. In 2016, Ahmad et al. [2] obtained the edge irregularity strength of some classes of Toeplitz graphs.

The current article attends to obtain the exact value of $es(G)$ for certain sorts of graphs G. In the next section we list our used notations, in addition to some of recent results regarding to our calculations. Next, our results will be introduced and proved.

2 Terminologies and notations

The considered graphs in this article are undirected, connected and simple graphs. For a graph $G(V, E)$ and a vertex $u \in V$, the degree of v is the number of edges end in v. Let $|V(G)|$ and $|E(G)|$ denote the order and size of G, respectively. The maximum degree of G is $\Delta(G) = \max\{\deg(v) \mid v \in V\}$.

In this article, we study certain sorts of graphs. These graphs are path graph, cycle graph, and complete bipartite graph.

Let C_n and P_n denote cycle and path with n vertices, respectively. The bipartite graph G is the graph in which $V = V_1 \cup V_2$ and every edge $e \in E$ corresponds to two vertices, one from V_1 and the other in V_2. Let $|V_1| = m$ and $|V_2| = n$. If every vertex $v \in V_1$ adjacent to all $u \in V_2$, then the graph is called complete bipartite graph and denoted by $K_{m,n}$ or $K_{n,m}$.

Definition 2.1. *The corona product of two graphs G and H, denoted by $G \odot H$, is a graph obtained by taking one copy of G and n copies of H (i.e $H_1, H_2, \ldots, H_n$ where $H_i \cong H$, $\forall i = 1, \ldots n$), and then joining the i^{th} vertex of G to every vertex in H_i for all $i = 1, 2, \ldots, n$.*

The main topic in this article is the edge irregularity strength of a given graph G. The value of $es(G)$ can be considered by defining an irregular k-labeling map $\psi : V \to \{1, 2, \cdots, k\}$ in which the smallest such integer k is used. A preferable lower bound of the edge irregularity strength has been found in the next theorem.

Theorem 2.2. *[3] Let G be simple graph with size $m = |E(G)|$ and maximum degree $\Delta = \Delta(G)$. Then*

$$es(G) \geq \max\left\{\left\lceil \frac{m+1}{2} \right\rceil, \Delta\right\}$$

Recently, some researchers determined the exact value of the edge irregularity strength of some graphs. For instance, the exact value of the edge irregularity strength of caterpillars, n-star graphs, (n, t)-kite graphs, cycle chains and friendship graphs have been studied in [8]. The edge irregularity strength of corona product of graphs with paths was studied in [10]. Ahmad et al. [1] obtained the exact value of edge irregularity strength of some chain graphs and join of two graphs. For more results on exact value of edge irregularity strength of graphs, please refer [7, 10, 9]. Some conjectures and open problems are given in respective papers for further research.

The following result was obtained in [10].

Theorem 2.3. *[10] For any integer $n \geq 2$, $es(P_n \odot P_2) = 2n + 1$.*

Recently, Hasni et al. [7] extended Theorem 2 and obtained the following result.

Theorem 2.4. *[7] For any integer $n \geq 2$ and $m = 3, 4, 5$, $es(P_n \odot P_m) = nm + 1$.*

Tarawneh et al. [9] determined the exact value of edge irregularity strength of corona product of C_n with P_2 and P_3.

Theorem 2.5. *[9] For any integer $n \geq 4$, $es(C_n \odot P_2) = \lceil \frac{4n+1}{2} \rceil$.*

Theorem 2.6. *[9] For any integer $n \geq 4$, $es(C_n \odot P_3) = \lceil \frac{6n+1}{2} \rceil$.*

Motivated by the above results, in the next section we determine the exact value of edge irregularity strength of complete bipartite graph, corona product of P_n with P_6 and P_n with C_3. The method used in this paper is slightly different with the existing papers.

3 Main Results

In this section, our main results will be presented. First, we study the edge irregularity strength of the complete bipartite graph $K_{n,2}$.

Theorem 3.1. *For any natural number n, $es(K_{n,2}) = n + 1$.*

Proof.
Let $G = K_{n,2}$ where $V_1 = \{x_1, x_2, \ldots, x_n\}$ and $V_2 = \{y_1, y_2\}$. The edge set $E = \{x_i y_j \mid i = 1, 2, \ldots, n \ , \ j = 1, 2\}$. Define the k-labeling map $\phi : V_1 \cup V_2 \to \{1, 2, \ldots, n+1\}$ by $\phi(x_i) = i$, $\phi(y_1) = 1$ and $\phi(y_2) = n + 1$. Then ϕ is well defined and $\phi(x_i y_j) = \phi(x_i) + \phi(y_j)$ for $i = 1, 2, \ldots, n$ and $j = 1, 2$ are the edge weights of G, which are distinct. Therefore, ϕ is an edge irregular k-labeling, implies $es(G) \leq es_\phi(G) = n+1$. Since the maximal degree of G is $\Delta(G) = n$ and $|E| = 2n$, then $\lceil \frac{2n+1+1}{2} \rceil = n+1 > \Delta(G)$. Thus using Theorem 2.2 we have $es(G) \geq n + 1$. Hence, $es(G) = n + 1$.

We now consider the edge irregularity strength of corona product of P_n with P_6.

Theorem 3.2. *$es(P_n \odot P_6) = 6n + 1$ for all $n \geq 2$.*

Proof.
Let $G = P_n \odot P_6$, and $P_6^i \cong P_6$ for all $i = 1, 2, \ldots, n$. Let H_i be the subgraph of G that obtained by joining x_i with all vertices of P_6^i for each $x_i \in P_n$. Then $V(H_i) = \{x_i, y_1^i, y_2^i, y_3^i, y_4^i, y_5^i, y_6^i\}$ for all $i = 1, 2, \ldots, n$. Then, the order of H_i is $6+1 = 7$ and the size is $6-1+6 = 11$. Therefore, $|V| = n(6)+n = 7n \geq 14$ and $|E| = 5n + 6n + n - 1 = 12n - 1$. Clearly, $\left\lceil \dfrac{|E|+1}{2} \right\rceil = 6n \geq 12$. On the other hand, $\Delta(G) = \begin{cases} 6, n = 2 \\ 7, n > 2 \end{cases}$.
Using Theorem 2.2, we have

$$es(G) \geq \max\left\{ \left\lceil \frac{|E|+1}{2} \right\rceil, \Delta(G) \right\} = 6n.$$

For $n \geq 2$, then $|V| \geq 14$. So,

$$es(G) > 6n \tag{3.1}$$

Define the vertex k-labeling $\phi : V \to \{1, 2, 3, \ldots, k\}$, where $k = 6n+1$ by

$\{\phi(x) \mid x \in H_i\} = \{6i-2, 6i-4, 6i-5, 6i-3, 6i-1, 6i+1, 6i\}$ for all $i = 1, 2, \ldots, n$.

Then the vertex labels are

$$\begin{aligned} H_1 &: \{4,2,1,3,5,7,6\} \\ & \quad \text{where } \phi(x_1)=4 \\ H_2 &: \{10,8,7,9,11,13,12\} \\ & \quad \text{where } \phi(x_2)=(7+6)+1-\phi(x_1)=14-4=10 \\ &\vdots \\ H_n &: \{6n-5,6n-4,6n-3,6n-2,6n-1,6n,6n+1\} \\ & \quad \text{where } \phi(x_n)=(6(n-1)+1+6(n-1))+1-\phi(x_{n-1})=12n-10-\phi(x_{n-1}) \end{aligned}$$

The corresponding edge weights in H_i is $w_i=\{12i-9,12i-8,12i-7,12i-6,12i-5,12i-4,12i-3,12i-2,12i-1,12i,12i+1\}$, where $i=1,2,\ldots,n$. Certainly, the edges weights of each H_i are distinct. To show that there are no edges in different H_i have the same weight, suppose in the contrary that $e_i \in H_i$ and $e_j \in H_j$ have the same weight for $i \neq j$. Comparing the set of weights, we consider the following 11 cases.

- If $12i-9=12j-9$, then $i=j$, so $e_i=e_j$ and $H_i=H_j$. Which contradicts our assumption.
- If $12i-9=12j-8$, then $12(i-j)=1$, implies that $i,j \notin \mathbb{N}$, which leads to a contradiction. Similar results can be obtained for the other cases, for which we have $12(i-j)=k$, $k=2,3,\cdots,10$.

Certainly, all of the other possible choices give the same contributions. Finally, The weights corresponding to the edges of P_n are $\phi(x_ix_{i+1})=\phi(x_i)+\phi(x_{i+1})=\phi(x_i)+12i-10-\phi(x_i)=12i-10$ for all $i=1,2,\ldots,n-1$, and these weights are not in any w_i for all i. Thus, ϕ is an edge irregular k-labeling of G, which gives $es(G)\leq es_\phi(G)=\max\{\phi(x)\mid x\in V\}=6n+1$. Using Equation 3.1, implies that $6n+1\geq es(G)>6n$. Hence, $es(G)=6n+1$.

We now consider the edge irregularity strength of corona product of P_n with C_3.

Theorem 3.3. *$es(P_n \odot C_3)=\lceil \frac{7n}{2}\rceil+1$ for $n\geq 3$.*

Proof.
Let $G=P_n\odot C_3$. For the case of $n\geq 3$, the maximum degree of G is

$\Delta(G) = 5$. The size of G is $|E| = n - 1 + 3n + 3n = 7n - 1$. Using Theorem 2.2, we have

$$es(G) \geq \max\left\{\left\lceil\frac{7n-1+1}{2}\right\rceil, \Delta(G)\right\} = \max\left\{\left\lceil\frac{7n}{2}\right\rceil, 4\right\}.$$

The term $\lceil\frac{7n}{2}\rceil$ is increasing, and the smallest such value can be obtained when $n = 3$ is $\left\lceil\frac{7(3)}{2}\right\rceil = 11$. Actually, if $n = 3$, then $es(G) = 12 = 11 + 1 > 11 = \left\lceil\frac{7(3)}{2}\right\rceil$. Therefore:

$$es(G) \geq \left\lceil\frac{7n}{2}\right\rceil + 1 \tag{3.2}$$

Let $\{x_1, x_2, \ldots, x_n\}$ be the set of vertices of P_n and $\{y_1^i, y_2^i, y_3^i\}$ be the set of vertices of C_3^i. Let H_i be the subgraph of G of vertices set $V_i = \{x_i, y_1^i, y_2^i, y_3^i\}$ and edges set $E_i = \{x_iy_1^i, x_iy_2^i, x_iy_3^i, y_1^iy_2^i, y_1^iy_3^i, y_2^iy_3^i\}$. Clearly, H_i is K_4 and the minimum edge weight is 3 for all $i = 1, 2, \ldots, n$.

Define the vertex k-labeling $\phi : V \to \{1, 2, 3, \ldots, k\}$, where $k = \lceil\frac{7n}{2}\rceil + 1$ as follows

$$\{\phi(x) \mid x \in H_1\} = \{\phi(y_1^1) = 1, \phi(y_2^1) = 2, \phi(y_3^1) = 3, \phi(x_1) = 5\}$$

$$\{\phi(x) \mid x \in H_2\} = \{\phi(x_2) = 4, \phi(y_1^2) = 6, \phi(y_2^2) = 7, \phi(y_3^2) = 8\}$$

and for all $i = 3, 4, \ldots, n$. Let $\phi(x_i) = \phi(x_{i-2}) + 7$ and $\phi(y_j^i) = \phi(y_j^{i-2}) + 7$ for $j = 1, 2, 3$.

Thus, the labeling set of H_n is
for n is odd, $\phi(y_1^n) = 7\left(\frac{n-1}{2}\right) + 1$, $\phi(y_2^n) = 7\left(\frac{n-1}{2}\right) + 2$, $\phi(y_3^n) = 7\left(\frac{n-1}{2}\right) + 3$ and $\phi(x_n) = 7\left(\frac{n-1}{2}\right) + 5$, and
for n is even, $\phi(x_n) = 7\left(\frac{n-2}{2}\right) + 4$, $\phi(y_1^n) = 7\left(\frac{n-2}{2}\right) + 6$, $\phi(y_2^n) = 7\left(\frac{n-2}{2}\right) + 7$ and $\phi(y_3^n) = 7\left(\frac{n-2}{2}\right) + 8$.

Therefore,

$$es_\phi(G) = \begin{cases} \frac{7n+3}{2}, n \text{ is odd} \\ \\ \frac{7n}{2} + 1, n \text{ is even} \end{cases}$$

Without lose of generality, assume that $i \in \{1, 2, \ldots, n\}$ is odd. Let $Y = \{\phi(y_j^iy_m^i) \mid j \neq m = 1, 2, 3\}$ be the set of all edges weights for C_3^i, and $XY = \{\phi(x_iy_j^i) \mid j = 1, 2, 3\}$ be the set of weights of the edges that joint x_i with all vertices in C_3^i. Finally, let $X = \{\phi(x_ix_{i+1}) \mid i = 1, 2, \ldots, n-1\}$

be the set of all edges weights that joint H_i and H_{i+1}. Clearly, the set of all possible edge weights for the case i is odd is $W = \{7i-4, 7i-3, 7i-2, 7i-1, 7i, 7i+1, 7i+2\}$, which all are distinct and increasing, so that none of these weights is repeating. Similarly, the same conclusion obtained for i is even. Therefore, ϕ is an edge irregular k-labeling of G. So, $es(G) \leq \lceil \frac{7n}{2} \rceil + 1$. Using Equation 3.2, implies that $es(G) = \lceil \frac{7n}{2} \rceil + 1$.

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